

VOEVODSKY'S MIXED MOTIVES VERSUS KONTSEVICH'S NONCOMMUTATIVE MIXED MOTIVES

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ABSTRACT. Following an insight of Kontsevich, we prove that the quotient of Voevodsky's category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}$ by the endofunctor $-\otimes \mathbb{Q}(1)[2]$ embeds fully-faithfully into Kontsevich's category of noncommutative mixed motives KMM. We show also that this embedding is compatible with the one between pure motives. As an application, we obtain a precise relation between the Picard groups $\mathrm{Pic}(-)$, the Grothendieck groups, the Schur-finiteness, and the Kimura-finiteness of the categories $\mathrm{DM}_{\mathrm{gm}}$ and KMM. In particular, the quotient of $\mathrm{Pic}(\mathrm{DM}_{\mathrm{gm}})$ by the subgroup of Tate twists $\mathbb{Q}(i)[2i]$ injects into $\mathrm{Pic}(\mathrm{KMM})$. Along the way, we relate KMM with Morel-Voevodsky's stable A^1 -homotopy category, recover the twisted algebraic K -theory of Kahn-Levine from KMM, and extend Elmendorf-Mandell's foundational work on multicategories to a broader setting.

CONTENTS

1. Introduction	1
2. Statement of results	2
3. Applications	5
4. Preliminaries	7
5. Noncommutative motives	8
6. Algebras over multicategories	10
7. Commutative monoids in symmetric spectra	16
8. A key commutative monoid	16
9. Proof of the first main result	22
10. Proof of the second main result	27
11. Remaining proofs	28
References	31

1. INTRODUCTION

Voevodsky's mixed motives. V. Voevodsky introduced in [62, §5] the triangulated category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k)$ (over a perfect base field k). This category comes equipped with a canonical functor $M : \mathrm{Sm}(k) \rightarrow \mathrm{DM}_{\mathrm{gm}}(k)$, defined on smooth k -schemes, and is the natural setting for the study of algebraic cycle (co)homology theories such as higher Chow groups, Suslin homology, motivic cohomology, bivariant cycle cohomology, etc.

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Kontsevich’s noncommutative mixed motives. M. Kontsevich introduced in [31] the triangulated category of noncommutative mixed motives $\mathrm{KMM}(k)$ (over a base field k). Roughly speaking, $\mathrm{KMM}(k)$ is the thick triangulated envelope of a category whose objects are the smooth proper dg categories (see Definition 4.2) and whose morphisms are given by bivariant algebraic K -theory. As explained in §5.2, Kontsevich’s category admits a more conceptual description: the smooth proper dg categories are the strongly dualizable objects of the Morita homotopy category of dg categories $\mathrm{Ho}(\mathrm{dgc}at(k))$; there exists a “universal” localizing invariant $U : \mathrm{Ho}(\mathrm{dgc}at(k)) \rightarrow \mathrm{Mot}(k)$ with values in a triangulated category; and $\mathrm{KMM}(k)$ identifies with the thick triangulated subcategory of $\mathrm{Mot}(k)$ generated by the images of the strongly dualizable objects. For this reason, $\mathrm{KMM}(k)$ is the natural setting for the study of localizing invariants such as algebraic K -theory, cyclic homology, topological Hochschild homology, etc; consult the survey [51].

Motivating question. As explained above, the triangulated categories $\mathrm{DM}_{\mathrm{gm}}(k)$ and $\mathrm{KMM}(k)$ play a similar role, one in the commutative world and the other one in the noncommutative world. Hence, it is natural to ask the following:

Question: What is the relation between $\mathrm{DM}_{\mathrm{gm}}(k)$ and $\mathrm{KMM}(k)$?

Kontsevich suggested in [33, §4.1] that the quotient of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ by the endofunctor $- \otimes \mathbb{Q}(1)[2]$ should embed fully-faithfully into $\mathrm{KMM}(k)_{\mathbb{Q}}$. In this article we answer the above motivating question and as a byproduct prove Kontsevich’s insight. Consult §3 for several applications of these results.

2. STATEMENT OF RESULTS

Let k be a perfect base field, $\mathrm{SH}(k)$ the Morel-Voevodsky stable A^1 -homotopy category of (\mathbb{P}^1, ∞) -spectra (see [38, 39, 60]), and $\mathrm{KGL} \in \mathrm{Ho}(\mathrm{SH}(k))$ the E^∞ -ring spectrum representing homotopy algebraic K -theory in the sense of Weibel (see [60, §6.2]). Thanks to the work of Röding-Sptizweck-Østvær [44] and Gepner-Snaith [17], KGL admits a strictly commutative model. Hence, we obtain a well-defined symmetric monoidal model category $\mathrm{Mod}(\mathrm{KGL})$ of KGL -modules. Let us denote by $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ the thick triangulated subcategory of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))$ generated by the objects $\Sigma^\infty(X_+) \wedge \mathrm{KGL}$ with X a smooth projective k -scheme.

As explained in §5.3, there exists also a “universal” A^1 -localizing invariant $U_{A^1} : \mathrm{Ho}(\mathrm{dgc}at(k)) \rightarrow \mathrm{Mot}_{A^1}(k)$. The category $\mathrm{Mot}_{A^1}(k)$ carries a closed symmetric monoidal structure (we write $(-)^{\vee} := \underline{\mathrm{Hom}}(-, U_{A^1}(k))$) making U_{A^1} symmetric monoidal. Motivated by the above description of $\mathrm{KMM}(k)$, we define $\mathrm{KMM}_{A^1}(k)$ to be the thick triangulated subcategory of $\mathrm{Mot}_{A^1}(k)$ generated by the objects $U_{A^1}(\mathcal{A})$ with \mathcal{A} a smooth proper dg category. Note that since the smooth proper dg categories are strongly dualizable objects, $\mathrm{KMM}_{A^1}(k)$ is a rigid symmetric monoidal category. The smallest triangulated subcategory of $\mathrm{Mot}(k)$ (resp. of $\mathrm{Mot}_{A^1}(k)$) which contains $\mathrm{KMM}(k)$ (resp. $\mathrm{KMM}_{A^1}(k)$) and is stable under arbitrary direct sums will be denoted by $\mathrm{KMM}^{\oplus}(k)$ (resp. by $\mathrm{KMM}_{A^1}^{\oplus}(k)$).

As explained in §4.3, the derived category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k -scheme X admits a canonical dg enhancement¹ $\mathrm{perf}_{\mathrm{dg}}(X)$. The first main result, obtained in collaboration with Denis-Charles Cisinski, is the following:

¹When X is quasi-projective, Lunts-Orlov proved in [36] that this dg enhancement is “unique”.

Theorem 2.1. *There exists a well-defined triangulated comparison functor Φ from $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))$ to $\mathrm{Mot}_{A^1}(k)$ and a natural transformation θ from*

$$(2.2) \quad \mathrm{Sm}(k) \xrightarrow{\Sigma^\infty(-)_+} \mathrm{Ho}(\mathrm{SH}(k)) \xrightarrow{-\wedge^{\mathrm{KGL}}} \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL})) \xrightarrow{\Phi} \mathrm{Mot}_{A^1}(k)$$

to the composition (the first and last functors are contravariant)

$$(2.3) \quad \mathrm{Sm}(k) \xrightarrow{\mathrm{perf}_{\mathrm{dg}}(-)} \mathrm{Ho}(\mathrm{dgc}at(k)) \xrightarrow{U_{A^1}} \mathrm{Mot}_{A^1}(k) \xrightarrow{(-)^\vee} \mathrm{Mot}_{A^1}(k).$$

This data has the following properties:

- (i) *The functor Φ is lax symmetric monoidal and preserves arbitrary direct sums;*
- (ii) *The functor Φ becomes symmetric monoidal and fully-faithful when restricted to the subcategory $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$;*
- (iii) *The natural transformation θ is an isomorphism at every smooth k -scheme X such that $\Sigma^\infty(X_+) \wedge \mathrm{KGL} \in \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$;*
- (iv) *The restriction of Φ to the subcategory $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ lifts along the composition $\mathrm{KMM}(k) \rightarrow \mathrm{KMM}_{A^1}(k) \subset \mathrm{Mot}_{A^1}(k)$.*

Intuitively speaking, Theorem 2.1 shows that the difference between the categories of Morel-Voevodsky and Kontsevich is measured by the existence of a KGL-module structure. The proof, envisioned by Cisinski, is divided into four steps:

- (s1) First, we extend Elmendorf-Mandell's foundational work on multicategories to a broader setting; see §6. This is of independent interest.
- (s2) Then, we establish a practical result for the construction of commutative monoids in generalized symmetric spectra; see §7.
- (s3) Making use of (s1)-(s2), we then construct a commutative monoid $\mathrm{KGL}_{\mathrm{nc}}$ in Ayoub's stable A^1 -homotopy category of (\mathbb{P}^1, ∞) -spectra (with coefficients in noncommutative mixed motives) which enhances KGL; see §8.
- (s4) Finally, making use of $\mathrm{KGL}_{\mathrm{nc}}$ and of the functoriality of Ayoub's stable A^1 -homotopy category of (\mathbb{P}^1, ∞) -spectra, we obtain the above result; see §9.

Corollary 2.4. *(i) When k admits resolution of singularities (e.g. $\mathbb{Q} \subseteq k$), there exists a fully-faithful symmetric monoidal triangulated comparison functor Φ making the following diagram commute*

$$\begin{array}{ccc} \mathrm{Sm}(k) & \xrightarrow{\mathrm{perf}_{\mathrm{dg}}(-)} & \mathrm{Ho}(\mathrm{dgc}at(k)) \\ \Sigma^\infty(-)_+ \downarrow & & \downarrow U \\ \mathrm{Ho}(\mathrm{SH}(k)) & & \mathrm{Mot}(k) \\ -\wedge^{\mathrm{KGL}} \downarrow & & \downarrow (-)^\vee \\ \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL})) & \xrightarrow{\Phi} & \mathrm{KMM}^\oplus(k) \subset \mathrm{Mot}(k). \end{array}$$

(ii) When k is a perfect field, there exists a fully-faithful symmetric monoidal triangulated comparison functor $\Phi_{\mathbb{Q}}$ making the following diagram commute

$$\begin{array}{ccc} \mathrm{Sm}(k) & \xrightarrow{\mathrm{perf}_{\mathrm{dg}}(-)} & \mathrm{Ho}(\mathrm{dgc}at(k)) \\ \Sigma^\infty(-)_+_{\mathbb{Q}} \downarrow & & \downarrow U(-)_{\mathbb{Q}} \\ \mathrm{Ho}(\mathrm{SH}(k))_{\mathbb{Q}} & & \mathrm{Mot}(k)_{\mathbb{Q}} \\ -\wedge^{\mathrm{KGL}_{\mathbb{Q}}} \downarrow & & \downarrow (-)^\vee \\ \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathbb{Q}})) & \xrightarrow{\Phi_{\mathbb{Q}}} & \mathrm{KMM}^\oplus(k)_{\mathbb{Q}} \subset \mathrm{Mot}(k)_{\mathbb{Q}}. \end{array}$$

- (iii) The functors Φ and $\Phi_{\mathbb{Q}}$ admit right adjoints Ψ and $\Psi_{\mathbb{Q}}$.
 (iv) Given a central simple k -algebra A , one has an isomorphism $\Psi(U(A)) \simeq K^A$ in $\mathrm{Ho}(\mathrm{SH}(k))$, where K^A stands for the twisted form of algebraic K -theory introduced by Kahn-Levine in [25].

Note that the comparison functors Φ and $\Phi_{\mathbb{Q}}$ take values in Kontsevich's triangulated category of noncommutative mixed motives (with arbitrary direct sums). Item (iv) furnish us a conceptual characterization of Kahn-Levine's construction.

Remark 2.5. Theorem 2.1 and items (i)-(iii) of Corollary 2.4 hold more generally when k is a regular ring; see Remark 9.32.

Relation between $\mathrm{DM}_{\mathrm{gm}}(k)$ and $\mathrm{KMM}(k)$. Let $\mathrm{HZ} \in \mathrm{Ho}(\mathrm{SH}(k))$ be the E^{∞} -ring spectrum representing motivic cohomology; see [60, §6.1]. Thanks to the work of Riou [42, §6], one has $\mathrm{KGL}_{\mathbb{Q}} \simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{HZ}_{\mathbb{Q}}(i)[2i]$. On the other hand, thanks to the work of Röding-Ostvær [45], $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ identifies with the full subcategory of compact objects of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{HZ}_{\mathbb{Q}}))$. As a consequence, base-change along $\mathrm{HZ}_{\mathbb{Q}} \rightarrow \mathrm{KGL}_{\mathbb{Q}}$ gives rise to a well-defined functor $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \rightarrow \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathbb{Q}}))^{\mathrm{pj}}$. By composing it with $\Phi_{\mathbb{Q}}$ one then obtains a \mathbb{Q} -linear faithful symmetric monoidal triangulated comparison functor

$$(2.6) \quad \mathrm{R} : \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \longrightarrow \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathbb{Q}}))^{\mathrm{pj}} \xrightarrow{\Phi_{\mathbb{Q}}} \mathrm{KMM}(k)_{\mathbb{Q}}.$$

The second main result, which answers the motivating question, is the following:

Theorem 2.7. *The comparison functor (2.6) gives rise to a \mathbb{Q} -linear additive fully-faithful symmetric monoidal functor $\overline{\mathrm{R}}$ making the following diagram commute*

$$(2.8) \quad \begin{array}{ccc} \mathrm{Sm}(k) & \xrightarrow{\mathrm{perf}_{\mathrm{dg}}(-)} & \mathrm{Ho}(\mathrm{dgc}at(k)) \\ M(-)_{\mathbb{Q}} \downarrow & & \downarrow U(-)_{\mathbb{Q}} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} & & \mathrm{Mot}(k)_{\mathbb{Q}} \\ \pi \downarrow & & \downarrow (-)^{\vee} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} / - \otimes_{\mathbb{Q}(1)[2]} & \xrightarrow{\overline{\mathrm{R}}} & \mathrm{KMM}(k)_{\mathbb{Q}} \subset \mathrm{Mot}(k)_{\mathbb{Q}}, \end{array}$$

where $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} / - \otimes_{\mathbb{Q}(1)[2]}$ stands for the orbit category of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ with respect to the endofunctor $- \otimes_{\mathbb{Q}(1)[2]}$ (see §4.4).

Note that Theorem 2.7 formalizes Kontsevich's beautiful insight: the quotient of the commutative world by the endofunctor $- \otimes_{\mathbb{Q}(1)[2]}$ embeds fully-faithfully into the noncommutative world. This opens new horizons and opportunities for research by enabling the interchange of results, techniques, ideas, and insights between the commutative and noncommutative worlds; see §3.

Compatibility with pure motives. The pure analogue of Theorem 2.7 was established in [50, §1]. In that case, $M(-)_{\mathbb{Q}}$ is replaced by the classical (contravariant) functor to Chow motives $h(-)_{\mathbb{Q}} : \mathrm{SmProj}(k) \rightarrow \mathrm{Chow}(k)_{\mathbb{Q}}$, $U(-)_{\mathbb{Q}}$ by the “universal” additive invariant (see §5.4), $\mathrm{KMM}(k)_{\mathbb{Q}}$ by the additive category of noncommutative Chow motives $\mathrm{NChow}(k)_{\mathbb{Q}}$, and $\overline{\mathrm{R}}$ by a \mathbb{Q} -linear fully-faithful symmetric monoidal functor $\overline{\mathrm{R}}$. Moreover, since $h(-)_{\mathbb{Q}}$ is contravariant, the functor $(-)^{\vee}$ is not used. The compatibility between $\overline{\mathrm{R}}$ and $\overline{\mathrm{R}}$ is the following:

Proposition 2.9. *There exists a \mathbb{Q} -linear additive fully-faithful symmetric monoidal functor V_{nc} making the following diagram commute*

$$(2.10) \quad \begin{array}{ccccccc} \text{SmProj}(k) & \xrightarrow{h(-)_{\mathbb{Q}}} & \text{Chow}(k)_{\mathbb{Q}} & \xrightarrow{\pi} & \text{Chow}(k)_{\mathbb{Q}} / -_{\otimes \mathbb{Q}(1)} & \xrightarrow{\overline{R}} & \text{NChow}(k)_{\mathbb{Q}} \\ \downarrow & & \downarrow V & & \downarrow \overline{V} & & \downarrow V_{\text{nc}} \\ & & \text{DM}_{\text{gm}}(k)_{\mathbb{Q}} & \xrightarrow{\pi} & \text{DM}_{\text{gm}}(k)_{\mathbb{Q}} / -_{\otimes \mathbb{Q}(1)[2]} & \xrightarrow{\overline{R}} & \text{KMM}(k)_{\mathbb{Q}} \\ \text{Sm}(k) & \xrightarrow{M(-)_{\mathbb{Q}}} & & & & & \downarrow (-)^{\vee} \\ & & & & & & \text{KMM}(k)_{\mathbb{Q}} \end{array}$$

where V stands for the \mathbb{Q} -linear additive fully-faithful (contravariant) functor constructed by Voevodsky in [62, §4] and \overline{V} its extension to the orbit categories.

Note that the functor \overline{V} is well-defined since $V(\mathbb{Q}(1)) \simeq \mathbb{Q}(-1)[-2]$. The (co-variant) functor V_{nc} is morally the noncommutative analogue of V .

3. APPLICATIONS

Picard groups. Given a symmetric monoidal category \mathcal{C} , its Picard group $\text{Pic}(\mathcal{C})$ is defined as the (abelian) group of isomorphism classes of \otimes -invertible objects.

Proposition 3.1. (i) *A geometric mixed motive M is \otimes -invertible if and only if the noncommutative mixed motive $R(M)$ is \otimes -invertible;*
(ii) *The comparison functor (2.6) induces an injective group homomorphism*

$$(3.2) \quad \text{Pic}(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}) / \{\mathbb{Q}(i)[2i] \mid i \in \mathbb{Z}\} \hookrightarrow \text{Pic}(\text{KMM}(k)_{\mathbb{Q}}).$$

Item (i) shows that the comparison functor (2.6) reflects \otimes -invertibility. On the other hand, item (ii) shows that two \otimes -invertible geometric mixed motives become isomorphic in the noncommutative world if and only if they are in the same orbit of the \mathbb{Z} -action $M \mapsto M(1)[2]$. The cokernel of (3.2) measures the existence of “truly \otimes -invertible noncommutative mixed motives”.

Example 3.3. Tobias proved in [59, §3] that the reduced geometric mixed motives $\widetilde{M}(\text{Spec}(l))_{\mathbb{Q}}(i)[j]$ (with $i, j \in \mathbb{Z}$ and l/k a field extension of degree ≤ 2) are \otimes -invertible. Making use of Proposition 3.1(ii), one then obtains the following subgroup of \otimes -invertible objects

$$(3.4) \quad \{\widetilde{U}(l)_{\mathbb{Q}}[j] \mid j \in \mathbb{Z} \text{ and } l/k \text{ with } [l : k] \leq 2\} \subset \text{Pic}(\text{KMM}(k)_{\mathbb{Q}}).$$

The left-hand-side of (3.4) identifies with the group $\mathbb{Z} \oplus k^{\times} / (k^{\times})^2$ when k is of characteristic $\neq 2$ and with $\mathbb{Z} \oplus k\{u + u^2 \mid u \in k\}$ when k is of characteristic 2.

Mixed Tate motives. Following Levine [34], let $\text{DMT}(k)_{\mathbb{Q}}$ be the thick triangulated subcategory of $\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$ generated by the objects $\mathbb{Q}(n), n \in \mathbb{Z}$. Since $R(\mathbb{Q}(n)) \simeq U(k)_{\mathbb{Q}}[-2n]$, the comparison functor (2.6) restricts to a \mathbb{Q} -linear faithful symmetric monoidal triangulated functor

$$(3.5) \quad R : \text{DMT}(k)_{\mathbb{Q}} \longrightarrow \langle U(k)_{\mathbb{Q}} \rangle \subset \text{KMM}(k)_{\mathbb{Q}}$$

with values in the thick triangulated subcategory generated by the \otimes -unit $U(k)_{\mathbb{Q}}$.

Example 3.6 (Kummer motives). Let k be a number field. Recall from [2, §20.3] that a 1-motive of the form $[\mathbb{Z} \xrightarrow{1 \mapsto q} \mathbb{G}_m]$, with $q \in k^\times$, is called a Kummer motive. Since these are extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$, we obtain the distinguished triangles

$$U(k)_\mathbb{Q}[-2] \longrightarrow R([\mathbb{Z} \xrightarrow{1 \mapsto q} \mathbb{G}_m]) \longrightarrow U(k)_\mathbb{Q} \longrightarrow U(k)_\mathbb{Q}[-1].$$

Remark 3.7. Recall from §5.2 that the morphisms of $\mathrm{KMM}(k)_\mathbb{Q}$ are given by algebraic K -theory. Hence, one observes that the maps induced by (3.5)

$$\mathrm{Hom}_{\mathrm{DMT}(k)_\mathbb{Q}}(\mathbb{Q}(0), \mathbb{Q}(n)[n]) \longrightarrow \mathrm{Hom}_{\langle U(k)_\mathbb{Q} \rangle}(U(k)_\mathbb{Q}, U(k)_\mathbb{Q}[-n]) \quad n \geq 0$$

correspond to the canonical inclusion of rational Milnor K -theory into rational (nonconnective) algebraic K -theory $K_n^M(k)_\mathbb{Q} \hookrightarrow K_n(k)_\mathbb{Q}$.

Grothendieck groups. The computation of the Grothendieck groups of $\mathrm{DM}_{\mathrm{gm}}(k)_\mathbb{Q}$ and $\mathrm{KMM}(k)_\mathbb{Q}$ is a major challenge which seems completely out of reach at the present time. In what concerns mixed Tate motives, Biglari proved in [5] that the assignment $\mathbb{Q}(1) \mapsto t$ gives rise to a ring isomorphism $K_0(\mathrm{DMT}(k)_\mathbb{Q}) \simeq \mathbb{Z}[t, t^{-1}]$.

Proposition 3.8. (i) *One has a ring isomorphism $K_0(\langle U(k)_\mathbb{Q} \rangle) \simeq \mathbb{Z}$;*
(ii) *The comparison functor (3.5) induces the following ring homomorphism*

$$(3.9) \quad K_0(\mathrm{DMT}(k)_\mathbb{Q}) \simeq \mathbb{Z}[t, t^{-1}] \xrightarrow{t=1} \mathbb{Z} \simeq K_0(\langle U(k)_\mathbb{Q} \rangle).$$

Intuitively speaking, Proposition 3.8 shows that “virtually” all the mixed Tate motives $\mathbb{Q}(n)$ become trivial in the noncommutative world. Note that in the case of Kummer motives, (3.9) corresponds to the passage from $t+1$ to 2.

Schur and Kimura finiteness. Let \mathcal{C} be a \mathbb{Q} -linear idempotent complete symmetric monoidal category (e.g. $\mathrm{DM}_{\mathrm{gm}}(k)_\mathbb{Q}$ or $\mathrm{KMM}(k)_\mathbb{Q}$). Every partition λ of n gives rise to an idempotent e_λ of the group ring $\mathbb{Q}[\Sigma_n]$ and hence to a Schur functor $S_\lambda : \mathcal{C} \rightarrow \mathcal{C}, c \mapsto e_\lambda(c^{\otimes n})$; consult Deligne [14, 15] for details. When $\lambda = (1, \dots, 1)$ (resp. $\lambda = (n)$) the associated Schur functor $\wedge^n := S_{(1, \dots, 1)}$ (resp. $\mathrm{Sym}^n := S_{(n)}$) should be considered the analogue of the usual n^{th} wedge (resp. symmetric) product of \mathbb{Q} -vector spaces. An object $c \in \mathcal{C}$ is called *Schur-finite* if $S_\lambda(c) = 0$ for some λ , *even* (resp. *odd*) *dimensional* if $\wedge^n(c) = 0$ (resp. $\mathrm{Sym}^n(c) = 0$) for some $n > 0$, and *Kimura-finite* if $c = c_+ \oplus c_-$ with c_+ even dimensional and c_- odd dimensional. In the particular case where $\mathcal{C} = \mathrm{DM}_{\mathrm{gm}}(k)_\mathbb{Q}$, these finiteness notions were extensively studied by André, Kahn, Guletskii, Pedrini, Kimura, and Mazza; see [2, 3, 19, 20, 29, 37].

Proposition 3.10. *Let M be a geometric mixed motive.*

- (i) *M is Schur-finite if and only if $R(M)$ is Schur-finite;*
- (ii) *If M is Kimura-finite, then $R(M)$ is also Kimura-finite.*

Item (i) shows that the comparison functor (2.6) reflects Schur-finiteness. On the other hand, item (ii) combined with the stability of Kimura-finiteness under several constructions (see [19, 37]), gives rise to a large class of Kimura-finite noncommutative mixed motives.

Example 3.11. Let n be an even positive integer and $u \neq 0$ an element of $K_n^M(k)_\mathbb{Q}$. As explained in §5.2, u can be understood as a morphism in $\mathrm{KMM}(k)_\mathbb{Q}$ from $U(k)_\mathbb{Q}$ to $U(k)_\mathbb{Q}[-n]$. Hence, consider the following distinguished triangle

$$U(k)_\mathbb{Q} \xrightarrow{u} U(k)_\mathbb{Q}[-n] \longrightarrow \mathrm{cone}(u) \longrightarrow U(k)_\mathbb{Q}[1].$$

By combining Biglari's work [5, Example 4.11] with Proposition 3.10, one observes that $\text{cone}(u)$ is Schur-finite but *not* even dimensional *neither* odd dimensional.

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4. PRELIMINARIES

4.1. Notations. Throughout the article we will work over a base field k . We will use freely the language of model categories; see [21, 23, 41]. Given a model category \mathcal{C} , we will write $\text{Ho}(\mathcal{C})$ for its homotopy category. The category of simplicial sets will be denoted by sSet , the category of spectra by Sp , the category of symmetric spectra (endowed with Hovey-Shipley-Smith's stable model structure [24]) by Sp^Σ , and the category of symmetric spectra (endowed with Shipley's stable positive model structure [48]) by \mathcal{S} . Given a closed symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, let $\underline{\text{Hom}}(-, -)$ be its internal Hom and $(-)^{\vee} := \underline{\text{Hom}}(-, \mathbf{1})$ the duality functor. When \mathcal{C} is enriched over a symmetric monoidal category \mathcal{E} we will write $\underline{\text{Hom}}_{\mathcal{E}}(-, -)$ for this enrichment. Finally, adjunctions will be displayed vertically with the left (resp. right) adjoint on the left (resp. right) hand-side.

4.2. Differential graded categories. Let $\mathcal{C}(k)$ be the category of cochain complexes of k -vector spaces. A *differential graded (=dg) category* \mathcal{A} is a category enriched over $\mathcal{C}(k)$. A *dg functor* $F : \mathcal{A} \rightarrow \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [26] for further details. In what follows, we will write $\text{dgc}at(k)$ for the category of (small) dg categories and dg functors.

Let \mathcal{A} be a dg category. The category $\text{H}^0(\mathcal{A})$ has the same objects as \mathcal{A} and $\text{H}^0(\mathcal{A})(x, y) := H^0\mathcal{A}(x, y)$. The opposite dg category \mathcal{A}^{op} has the same objects as \mathcal{A} and $\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)$. A *right \mathcal{A} -module* is a dg functor $\mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$ with values in the dg category $\mathcal{C}_{\text{dg}}(k)$ of complexes of k -vector spaces. Let us write $\mathcal{C}(\mathcal{A})$ for the category of right \mathcal{A} -modules. As explained in [26, §3.1], the dg structure of $\mathcal{C}_{\text{dg}}(k)$ makes $\mathcal{C}(\mathcal{A})$ into a dg category $\mathcal{C}_{\text{dg}}(\mathcal{A})$. The *derived category* $\mathcal{D}(\mathcal{A})$ of \mathcal{A} is the localization of $\mathcal{C}(\mathcal{A})$ with respect to the quasi-isomorphisms. Its subcategory of compact objects will be denoted by $\mathcal{D}_c(\mathcal{A})$.

A dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called a *Morita equivalence* if the restriction of scalars $\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$ is an equivalence. As proved in [54, Thm. 5.3], $\text{dgc}at(k)$ admits a model structure whose weak equivalences are the Morita equivalences.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A} \otimes \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$. As explained in [26, §2.3], this construction gives rise to symmetric monoidal categories $(\text{dgc}at(k), - \otimes -, k)$ and $(\text{Ho}(\text{dgc}at(k)), - \otimes -, k)$. Given dg categories \mathcal{A} and \mathcal{B} , an *\mathcal{A} - \mathcal{B} -bimodule* \mathcal{B} is a dg functor $\mathcal{B} : \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$, i.e. a right $(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$ -module. A standard example is the \mathcal{A} - \mathcal{A} -bimodule

$$(4.1) \quad \mathcal{A} \otimes \mathcal{A}^{\text{op}} \longrightarrow \mathcal{C}_{\text{dg}}(k) \quad (x, y) \mapsto \mathcal{A}(y, x).$$

Given dg categories \mathcal{A} and \mathcal{B} , let $\text{rep}(\mathcal{A}, \mathcal{B})$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$ consisting of those \mathcal{A} - \mathcal{B} -bimodules B such that $B(x, -) \in \mathcal{D}_c(\mathcal{B})$ for every object $x \in \mathcal{A}$. In the same vein, let $\text{rep}_{\text{dg}}(\mathcal{A}, \mathcal{B})$ be the full dg subcategory of $\mathcal{C}_{\text{dg}}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$ consisting of those \mathcal{A} - \mathcal{B} -bimodules B which belong to $\text{rep}(\mathcal{A}, \mathcal{B})$. By construction, we have $H^0(\text{rep}_{\text{dg}}(\mathcal{A}, \mathcal{B})) \simeq \text{rep}(\mathcal{A}, \mathcal{B})$.

Definition 4.2 (Kontsevich [31, 32, 33]). A dg category \mathcal{A} is called *smooth* if the above \mathcal{A} - \mathcal{A} -bimodule (4.1) belongs to $\mathcal{D}_c(\mathcal{A}^{\text{op}} \otimes \mathcal{A})$ and *proper* if for each ordered pair of objects (x, y) we have $\sum_i \dim H^i \mathcal{A}(x, y) < \infty$.

Remark 4.3. As proved in [8, Thm. 5.8], the smooth and proper dg categories can be characterized as being precisely the strongly dualizable objects of $\text{Ho}(\text{dgcat}(k))$.

4.3. Perfect complexes on schemes. Given a quasi-compact quasi-separated k -scheme X , let $\text{Mod}(X)$ be the Grothendieck category of sheaves of \mathcal{O}_X -modules, $\mathcal{D}(X) := \mathcal{D}(\text{Mod}(X))$ the derived category of X , and $\text{perf}(X) \subset \mathcal{D}(X)$ the full triangulated subcategory of perfect complexes; see Thomason-Trobaugh [58, §2].

As explained in [26, §4.4], the derived dg category $\mathcal{D}_{\text{dg}}(\mathcal{E})$ of an abelian category \mathcal{E} is defined as the dg quotient $\mathcal{C}_{\text{dg}}(\mathcal{E})/\mathcal{A}c_{\text{dg}}(\mathcal{E})$ of the dg category of complexes over \mathcal{E} by its full dg subcategory of acyclic complexes. Hence, let $\mathcal{D}_{\text{dg}}(X)$ be the dg category $\mathcal{D}_{\text{dg}}(\mathcal{E})$, with $\mathcal{E} := \text{Mod}(X)$, and $\text{perf}_{\text{dg}}(X) \subset \mathcal{D}_{\text{dg}}(X)$ the full dg subcategory of perfect complexes. By construction, we have $H^0(\mathcal{D}_{\text{dg}}(X)) \simeq \mathcal{D}(X)$ and $H^0(\text{perf}_{\text{dg}}(X)) \simeq \text{perf}(X)$. When X is smooth proper, the dg category $\text{perf}_{\text{dg}}(X)$ is smooth proper in the sense of Definition 4.2.

Finally, we will write $\text{Sm}(k)$ for the category of smooth k -schemes, $\text{Sm}'(k)$ for the category of quasi-compact smooth k -schemes, and $\text{SmProj}(k)$ for the category of smooth projective k -schemes.

4.4. Orbit categories. Let \mathcal{C} be a symmetric monoidal category and $O \in \mathcal{C}$ a \otimes -invertible object. Recall from [50, §7] the construction of the orbit category $\mathcal{C}/_{\otimes O}$. It has the same objects as \mathcal{C} and morphisms given by

$$\text{Hom}_{\mathcal{C}/_{\otimes O}}(a, b) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b \otimes O^{\otimes i}).$$

Given objects a, b, c and morphisms

$$\underline{f} = \{f_i\}_{i \in \mathbb{Z}} \in \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b \otimes O^{\otimes i}) \quad \underline{g} = \{g_i\}_{i \in \mathbb{Z}} \in \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(b, c \otimes O^{\otimes i}),$$

the i^{th} -component of composition $\underline{g} \circ \underline{f}$ is the finite sum $\sum_i (g_{-i} \otimes O^{\otimes i}) \circ f_i$. These definitions give rise to an additive category and to a canonical additive projection functor $\pi : \mathcal{C} \rightarrow \mathcal{C}/_{\otimes O}$. As proved in [50, Lem. 7.3], the orbit category inherits from \mathcal{C} a symmetric monoidal structure making π symmetric monoidal. Moreover, the functor π comes equipped with a natural 2-isomorphism $\pi \circ (- \otimes O) \xrightarrow{\sim} \pi$ and is 2-universal among all such functors.

5. NONCOMMUTATIVE MOTIVES

5.1. Grothendieck derivators. The theory of derivators allows us to state and prove precise universal properties. The original reference is Grothendieck's manuscript [18]; consult the Appendices of [7, 8] for shorter and more didactic accounts. Roughly speaking, a derivator \mathbb{D} consists of a strict contravariant 2-functor from the 2-category of small categories to the 2-category of all categories

$$\mathbb{D} : \text{Cat}^{\text{op}} \longrightarrow \text{CAT} \quad I \mapsto \mathbb{D}(I)$$

subject to several natural axioms. The essential example to keep in mind is the derivator $\mathbb{D} = \mathrm{HO}(\mathcal{C})$ associated to a model category \mathcal{C} and defined for every small category I by $\mathrm{HO}(\mathcal{C})(I) := \mathrm{Ho}(\mathrm{Fun}(I^{\mathrm{op}}, \mathcal{C}))$. Let e be the 1-point category with only one object and one identity morphism. By definition, $\mathbb{D}(e)$ is called the *base category* of the derivator \mathbb{D} . Heuristically, it is the basic “derived” category under consideration. For instance, if $\mathbb{D} = \mathrm{HO}(\mathcal{C})$ then $\mathbb{D}(e) = \mathrm{Ho}(\mathcal{C})$.

A derivator \mathbb{D} is called *triangulated* if $\mathbb{D}(I)$ is a triangulated category for every small category I . For example, the derivator $\mathrm{HO}(\mathcal{C})$ associated to a stable model category \mathcal{C} is triangulated. As explained in [7, §A.3], every triangulated derivator \mathbb{D} is naturally enriched $\underline{\mathrm{Hom}}_{\mathrm{Sp}}(-, -)$ over spectra.

5.2. Noncommutative mixed motives. As mentioned in §4.2, $\mathrm{dgc}at(k)$ carries a model structure. Consequently, one obtains a derivator $\mathrm{HO}(\mathrm{dgc}at(k))$. A morphism of derivators $E : \mathrm{HO}(\mathrm{dgc}at(k)) \rightarrow \mathbb{D}$, with values in a triangulated derivator, is called a *localizing invariant* (see [53, §10]) if it preserves filtered homotopy colimits and sends (Drinfeld) short exact sequences of dg categories to triangles

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A}'' \longrightarrow 0 \quad \mapsto \quad E(\mathcal{A}) \longrightarrow E(\mathcal{A}') \longrightarrow E(\mathcal{A}'') \longrightarrow E(\mathcal{A})[1].$$

Thanks to the work of Blumberg-Mandell, Keller, Schlichting, Thomason-Trobaugh, and others (see [4, 27, 46, 52, 58]), nonconnective algebraic K -theory (\mathbb{K}), cyclic homology, topological Hochschild homology, etc, give rise to localizing invariants. In [53, Def. 10.2] the universal localizing invariant $\mathcal{U} : \mathrm{HO}(\mathrm{dgc}at(k)) \rightarrow \mathcal{M}ot(k)$ was constructed. Given any triangulated derivator \mathbb{D} one has an induced equivalence

$$\mathcal{U}^* : \mathrm{HOM}_! (\mathcal{M}ot(k), \mathbb{D}) \xrightarrow{\sim} \mathrm{HOM}_{\mathrm{loc}} (\mathrm{HO}(\mathrm{dgc}at(k)), \mathbb{D}),$$

where the left-hand-side denotes the category of homotopy colimit preserving morphisms of derivators and the right-hand-side the category of localizing invariants. Moreover, as proved in [8, Thm. 8.5], the derivator $\mathcal{M}ot(k)$ carries a symmetric monoidal structure making \mathcal{U} symmetric monoidal. Let us denote by $U : \mathrm{Ho}(\mathrm{dgc}at(k)) \rightarrow \mathcal{M}ot(k)$ the restriction of \mathcal{U} to the base category. As proved in [8, Prop. 9.5], Kontsevich’s category of noncommutative mixed motives $\mathrm{KMM}(k)$ identifies with the thick triangulated subcategory of $\mathcal{M}ot(k)$ generated by the objects $U(\mathcal{A})$ with \mathcal{A} a smooth proper dg category. Note that thanks to Remark 4.3, $\mathrm{KMM}(k)$ is a rigid symmetric monoidal category. Recall from [8, Prop. 9.3] that given a smooth proper k -scheme X and quasi-compact quasi-separated k -scheme Y , one has a weak equivalence of spectra

$$(5.1) \quad \underline{\mathrm{Hom}}_{\mathrm{Sp}}(U(\mathrm{perf}_{\mathrm{dg}}(X)), U(\mathrm{perf}_{\mathrm{dg}}(Y))) \simeq \mathbb{K}(X \times Y).$$

5.3. A^1 -homotopy noncommutative mixed motives. A morphism of derivators $E : \mathrm{HO}(\mathrm{dgc}at(k)) \rightarrow \mathbb{D}$ is called *A^1 -homotopy invariant* (see [55, §1]) if it inverts the dg functors $\mathcal{A} \rightarrow \mathcal{A}[t] := \mathcal{A} \otimes k[t]$. A localizing invariant which is moreover A^1 -homotopy invariant is called an *A^1 -localizing invariant*. Thanks to the work of Thomason [57] and Weibel [63], homotopy algebraic K -theory (KH) and étale K -theory with finite coefficients give rise to A^1 -localizing invariants; consult [55, §5] for details. In [55, Thm. 2.1] the universal A^1 -localizing invariant $\mathcal{U}_{A^1} : \mathrm{HO}(\mathrm{dgc}at(k)) \rightarrow \mathcal{M}ot_{A^1}(k)$ was constructed. Given any triangulated derivator \mathbb{D} one has an induced equivalence

$$(\mathcal{U}_{A^1})^* : \mathrm{HOM}_! (\mathcal{M}ot_{A^1}(k), \mathbb{D}) \xrightarrow{\sim} \mathrm{HOM}_{A^1\text{-loc}} (\mathrm{HO}(\mathrm{dgc}at(k)), \mathbb{D}),$$

where the right-hand-side denotes the category of A^1 -localizing invariants. Moreover, the derivator $\mathcal{M}ot_{A^1}(k)$ carries a closed symmetric monoidal structure making \mathcal{U}_{A^1} symmetric monoidal. Let us denote by $U_{A^1} : \mathrm{Ho}(\mathrm{dgc}at(k)) \rightarrow \mathcal{M}ot_{A^1}(k)$ the restriction of \mathcal{U}_{A^1} to the base category. Recall from [55, Cor. 2.7] that given a smooth proper k -scheme X and quasi-compact quasi-separated k -scheme Y , one has a weak equivalence of spectra

$$(5.2) \quad \underline{\mathrm{Hom}}_{\mathrm{Sp}}(U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X)), U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(Y))) \simeq KH(X \times Y).$$

Remark 5.3. As proved in [49, Prop. 8.2], one has a natural isomorphism

$$U_{A^1}(- \boxtimes -) : U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X) \otimes \mathrm{perf}_{\mathrm{dg}}(Y)) \xrightarrow{\sim} U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X \times Y))$$

for any two quasi-compact quasi-separated k -schemes X and Y .

5.4. Noncommutative Chow motives. Given a dg category \mathcal{A} , let $T(\mathcal{A})$ be the dg category of pairs (i, x) , where $i \in \{1, 2\}$ and $x \in \mathcal{A}$. The complex of morphisms in $T(\mathcal{A})$ from (i, x) to (i', x') is given by $\mathcal{A}(x, x')$ if $i \leq i'$ and is zero otherwise. Note that we have two inclusion dg functors $i_1, i_2 : \mathcal{A} \hookrightarrow T(\mathcal{A})$. A functor $E : \mathrm{dgc}at(k) \rightarrow \mathbf{A}$, with values in an additive category, is called an *additive invariant* if it inverts the Morita equivalences and gives rise to the isomorphisms

$$[E(i_1) \ E(i_2)] : E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(T(\mathcal{A})).$$

In [54, §6] the universal additive invariant $U_{\mathrm{add}} : \mathrm{dgc}at(k) \rightarrow \mathrm{Hmo}_0(k)$ was constructed. Given any additive category \mathbf{A} one has an induced equivalence

$$(5.4) \quad (U_{\mathrm{add}})^* : \mathrm{Fun}_{\mathrm{additive}}(\mathrm{Hmo}_0(k), \mathbf{A}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgc}at(k), \mathbf{A}),$$

where the left-hand-side denotes the category of additive functors and the right-hand-side the category of additive invariants. Moreover, $\mathrm{Hmo}_0(k)$ carries a symmetric monoidal structure making U_{add} symmetric monoidal. The category of *noncommutative Chow motives* $\mathrm{NChow}(k)$ is defined as the additive subcategory of $\mathrm{Hmo}_0(k)$ generated by the objects $U_{\mathrm{add}}(\mathcal{A})$ with \mathcal{A} a smooth proper dg category. Note that thanks to Remark 4.3, $\mathrm{NChow}(k)$ is a rigid symmetric monoidal category.

6. ALGEBRAS OVER MULTICATEGORIES

This section is of independent interest. In Theorem 6.11 we establish an important “adjunction” formula and in Theorems 6.20 and 6.23 we extend Elmendorf-Mandell’s foundational work [16] to a broader setting. These general results will play a key role in the sequel. In what follows, \mathcal{E} is a (co)complete closed symmetric monoidal category (e.g. sSet) and Σ_n denotes the symmetric group of n symbols.

6.1. Multicategories. A *multicategory* \mathbf{M} consists of the following data:

- (i) A collection of objects $\mathrm{obj}(\mathbf{M})$;
- (ii) A set of “ n -morphisms” $\mathbf{M}_n(a_1, \dots, a_n; b)$ for each $n \geq 0$ and $(n+1)$ -tuple of objects $(a_1, \dots, a_n; b)$;
- (iii) A right action of Σ_n on the set of all “ n -morphisms”

$$\sigma^* : \mathbf{M}_n(a_1, \dots, a_n; b) \xrightarrow{\sim} \mathbf{M}_n(a_{\sigma(1)}, \dots, a_{\sigma(n)}; b) \quad \sigma \in \Sigma_n;$$

- (iv) A distinguished “unit” element $1_a \in \mathbf{M}_1(a; a)$ for each object a ;

(v) A composition “multiproduct”

$$\begin{array}{c} \mathbf{M}_k(b_1, \dots, b_k; c) \times \mathbf{M}_{n_1}(a_{11}, \dots, a_{1n_1}; b_1) \times \dots \times \mathbf{M}_{n_k}(a_{k1}, \dots, a_{kn_k}; b_k) \\ \downarrow \\ \mathbf{M}_{n_1 + \dots + n_k}(a_{11}, \dots, a_{kn_k}; c). \end{array}$$

This data is subject to natural axioms; see [16, Def. 2.1]. When the “ n -morphisms” belong to \mathcal{E} we say that \mathbf{M} is a \mathcal{E} -enriched multicategory. A multifunctor $h : \mathbf{M} \rightarrow \mathbf{M}'$ consists of a function $h : \text{obj}(\mathbf{M}) \rightarrow \text{obj}(\mathbf{M}')$ and maps

$$(6.1) \quad \mathbf{M}_n(a_1, \dots, a_n; b) \longrightarrow \mathbf{M}'_n(h(a_1), \dots, h(a_n); h(b))$$

preserving the Σ_n -action, the “unit” element, and the composition “multiproduct”; see [16, Def. 2.2]. When \mathbf{M} and \mathbf{M}' are \mathcal{E} -enriched multicategories and (6.1) is a map in \mathcal{E} , we say that h is a \mathcal{E} -enriched multifunctor.

Example 6.2 (Operads). An (\mathcal{E} -enriched) operad is the same data as an (\mathcal{E} -enriched) multicategory with a single object. An example is the operad Σ defined by the symmetric groups $\Sigma_n, n \geq 0$. By applying to it the symmetric monoidal functor $\mathcal{G} : \text{Set} \rightarrow \text{CAT}$ (which sends a set to the contractible groupoid with the same set of objects) we obtain a CAT-enriched operad $\mathcal{G}\Sigma$. Another example is the operad Comm characterized by having trivial “ n -morphisms” for every $n \geq 0$.

Example 6.3 (Symmetric monoidal categories). A symmetric monoidal category \mathcal{C} gives rise to a multicategory \mathcal{C}^\otimes : the objects are the same and $\mathcal{C}_n^\otimes(a_1, \dots, a_n; b) := \text{Hom}_{\mathcal{C}}(a_1 \otimes \dots \otimes a_n, b)$. When \mathcal{C} is enriched over \mathcal{E} , we obtain a \mathcal{E} -enriched multicategory \mathcal{C}^\otimes by setting $\mathcal{C}_n^\otimes(a_1, \dots, a_n; b) := \underline{\text{Hom}}_{\mathcal{E}}(a_1 \otimes \dots \otimes a_n, b)$. Every (\mathcal{E} -enriched) symmetric monoidal functor $\mathcal{C} \rightarrow \mathcal{C}'$ gives rise to a (\mathcal{E} -enriched) multifunctor $\mathcal{C}^\otimes \rightarrow (\mathcal{C}')^\otimes$.

Given multicategories \mathbf{M}, \mathbf{M}' , let $\mathbf{M} \times \mathbf{M}'$ be the multicategory with objects $\text{obj}(\mathbf{M}) \times \text{obj}(\mathbf{M}')$ and “ n -morphisms” given by

$$(\mathbf{M} \times \mathbf{M}')_n((a_1, a'_1), \dots, (a_n, a'_n); (b, b')) := \mathbf{M}_n(a_1, \dots, a_n; b) \times \mathbf{M}'_n(a'_1, \dots, a'_n; b').$$

When \mathbf{M} and \mathbf{M}' are \mathcal{E} -enriched multicategories we write $\mathbf{M} \otimes_{\mathcal{E}} \mathbf{M}'$ for the \mathcal{E} -enriched multicategory defined similarly but with \times replaced by $\otimes_{\mathcal{E}}$.

Definition 6.4. An \mathbf{M} -algebra in a symmetric monoidal category \mathcal{C} is a multifunctor $\alpha : \mathbf{M} \rightarrow \mathcal{C}^\otimes$. The category of \mathbf{M} -algebras in \mathcal{C} will be denoted by $\mathbf{M}\text{-Alg}(\mathcal{C}^\otimes)$. When \mathcal{C} is enriched over \mathcal{E} and \mathbf{M} is a \mathcal{E} -enriched multicategory, we will write $\mathbf{M}\text{-Alg}_{\mathcal{E}}(\mathcal{C}^\otimes)$ for the category of \mathcal{E} -enriched multifunctors $\mathbf{M} \rightarrow \mathcal{C}^\otimes$.

6.2. Day convolution product. Let \mathcal{D} be a small symmetric monoidal category, and $(\mathcal{C}, \otimes, \mathbf{1})$ a (co)complete closed symmetric monoidal category enriched over \mathcal{E} . Out of this data one can construct the category $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})$ of presheaves. Note that this latter category is enrichment over \mathcal{E} . Note also that we have a bifunctor

$$\mathcal{D} \times \mathcal{C} \longrightarrow \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C}) \quad (d, c) \mapsto d \otimes c := \coprod_{\text{Hom}_{\mathcal{D}}(-, d)} c.$$

Thanks to the \mathcal{E} -enriched Yoneda lemma, we have $\underline{\text{Hom}}_{\mathcal{E}}(d \otimes c, F) \simeq \underline{\text{Hom}}_{\mathcal{E}}(c, F(d))$ for every $F \in \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})$.

Definition 6.5. (Day [12, §3]) The *Day convolution product* $- \star -$ on $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})$ is the unique colimit preserving bifunctor which verifies $(d_1 \otimes c_1) \star (d_2 \otimes c_2) \simeq (d_1 \otimes d_2) \otimes (c_1 \otimes c_2)$ for every $d_1, d_2 \in \mathcal{D}$ and $c_1, c_2 \in \mathcal{C}$. As explained in *loc. cit.*,

this convolution product gives rise to a well-defined closed symmetric monoidal structure on $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})$.

Remark 6.6. Every $(\mathcal{E}$ -enriched) colimit preserving symmetric monoidal functor $\mathcal{C} \rightarrow \mathcal{C}'$ gives rise to a $(\mathcal{E}$ -enriched) multifunctor $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})^{\otimes} \rightarrow \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C}')^{\otimes}$.

Lemma 6.7. *Given $F_1, \dots, F_n, H \in \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})$, one has a natural isomorphism*

$$\underline{\text{Hom}}_{\mathcal{E}}(\star_{i=1}^n F_i, H) \simeq \lim_{\{c_i \rightarrow F_i(d_i)\}_{i=1}^n} \underline{\text{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n c_i, H(\otimes_{i=1}^n d_i)).$$

Proof. The proof follows from the natural isomorphisms

$$(6.8) \quad \underline{\text{Hom}}_{\mathcal{E}}(\star_{i=1}^n F_i, H) \simeq \underline{\text{Hom}}_{\mathcal{E}}(\star_{i=1}^n (\text{colim}_{d_i \otimes c_i \rightarrow F_i} d_i \otimes c_i), H)$$

$$(6.9) \quad \simeq \underline{\text{Hom}}_{\mathcal{E}}(\text{colim}_{\{d_i \otimes c_i \rightarrow F_i\}_{i=1}^n} ((\otimes_{i=1}^n d_i) \otimes (\otimes_{i=1}^n c_i)), H)$$

$$(6.10) \quad \simeq \lim_{\{c_i \rightarrow F_i(d_i)\}_{i=1}^n} \underline{\text{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n c_i, H(\otimes_{i=1}^n d_i)).$$

Some explanations are in order: (6.9) follows from the canonical presentation $\text{colim}_{d_i \otimes c_i \rightarrow F_i} d_i \otimes c_i \simeq F_i$; (6.9) follows from the properties of the Day convolution product; and (6.10) follows from the \mathcal{E} -enriched Yoneda lemma. \square

Theorem 6.11. *Given a \mathcal{E} -enriched multicategory \mathbf{M} , one has an equivalence*

$$(6.12) \quad \mathbf{M}\text{-Alg}_{\mathcal{E}}(\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})^{\otimes}) \xrightarrow{\sim} (\mathbf{M} \otimes_{\mathcal{E}} (\mathcal{D}^{\text{op}})^{\otimes})\text{-Alg}_{\mathcal{E}}(\mathcal{C}^{\otimes}) \quad \alpha \mapsto \bar{\alpha},$$

where \mathcal{D} is enriched over \mathcal{E} in the trivial way.

Proof. Recall first that by definition we have maps

$$(6.13) \quad \mathbf{M}_n(a_1, \dots, a_n; b) \longrightarrow \underline{\text{Hom}}_{\mathcal{E}}(\star_{i=1}^n \alpha(a_i), \alpha(b)).$$

We start by constructing the algebra $\bar{\alpha}$. On objects we set $(a, d) \mapsto \bar{\alpha}(a, d) := \alpha(a)(d)$. In order to define the maps

$$(6.14) \quad \mathbf{M}_n(a_1, \dots, a_n; b) \otimes_{\mathcal{E}} \underline{\text{Hom}}_{\mathcal{E}}(d, \otimes_{i=1}^n d_i) \longrightarrow \underline{\text{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n \bar{\alpha}(a_i, d_i), \bar{\alpha}(b, d)),$$

note that thanks to the \mathcal{C} -enriched Yoneda lemma we have

$$\begin{aligned} \otimes_{i=1}^n \underline{\text{Hom}}_{\mathcal{C}}(d_i \otimes \mathbf{1}, \alpha(a_i)) &\simeq \otimes_{i=1}^n \bar{\alpha}(a_i, d_i) \\ \underline{\text{Hom}}_{\mathcal{C}}((\otimes_{i=1}^n d_i) \otimes \mathbf{1}, \star_{i=1}^n \alpha(a_i)) &\simeq (\star_{i=1}^n \alpha(a_i))(\otimes_{i=1}^n d_i). \end{aligned}$$

Hence, by applying the functor $\underline{\text{Hom}}_{\mathcal{E}}(-, \bar{\alpha}(b, d))$ to

$$\otimes_{i=1}^n \underline{\text{Hom}}_{\mathcal{C}}(c_i \otimes \mathbf{1}, \alpha(a_i)) \longrightarrow \underline{\text{Hom}}_{\mathcal{C}}((\otimes_{i=1}^n d_i) \otimes \mathbf{1}, \star_{i=1}^n \alpha(a_i)),$$

we obtain the induced map

$$(6.15) \quad \underline{\text{Hom}}_{\mathcal{E}}((\star_{i=1}^n \alpha(a_i))(\otimes_{i=1}^n d_i), \bar{\alpha}(b, d)) \longrightarrow \underline{\text{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n \bar{\alpha}(a_i, d_i), \bar{\alpha}(b, d)).$$

The searched maps (6.14) are then defined by the following composition

$$\begin{aligned} &\mathbf{M}_n(a_1, \dots, a_n; b) \otimes_{\mathcal{E}} \underline{\text{Hom}}_{\mathcal{E}}(d, \otimes_{i=1}^n d_i) \\ &\quad \downarrow (6.13) \otimes_{\mathcal{E}} \text{id} \\ &\underline{\text{Hom}}_{\mathcal{E}}(\star_{i=1}^n \alpha(a_i); \alpha(b)) \otimes_{\mathcal{E}} \underline{\text{Hom}}_{\mathcal{E}}(d, \otimes_{i=1}^n d_i) \\ &\quad \downarrow \\ &\underline{\text{Hom}}_{\mathcal{E}}((\star_{i=1}^n \alpha(a_i))(\otimes_{i=1}^n d_i), \bar{\alpha}(b, d)) \\ &\quad \downarrow (6.15) \\ &\underline{\text{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n \bar{\alpha}(a_i, d_i), \bar{\alpha}(b, d)). \end{aligned}$$

This construction is functorial on α and gives rise to the above functor (6.12). Let us now construct its (quasi-)inverse $\beta \mapsto \bar{\beta}$. Recall that we have maps

$$\mathbf{M}_n(a_1, \dots, a_n; b) \otimes_{\mathcal{E}} \underline{\mathrm{Hom}}_{\mathcal{E}}(d, \otimes_{i=1}^n d_i) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n \beta(a_i, d_i), \beta(b, d)).$$

Since the \mathcal{E} -enrichment of \mathcal{D} is trivial, these maps reduce to

$$(6.16) \quad \coprod_{\mathrm{Hom}_{\mathcal{D}}(d, \otimes_{i=1}^n d_i)} (\mathbf{M}_n(a_1, \dots, a_n; b) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n \beta(a_i, d_i), \beta(b, d))).$$

Let us now construct $\bar{\beta}$. On objects we set $a \mapsto \bar{\beta}(a) := \beta(a, -)$. In order to define

$$(6.17) \quad \mathbf{M}_n(a_1, \dots, a_n; b) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{E}}(\star_{i=1}^n \bar{\beta}(a_i), \bar{\beta}(b)),$$

note that thanks to Lemma 6.7 (with $F_i = \bar{\beta}(a_i)$ and $H = \bar{\beta}(b)$) we have

$$\underline{\mathrm{Hom}}_{\mathcal{E}}(\star_{i=1}^n \bar{\beta}(a_i), \bar{\beta}(b)) \simeq \lim_{\{c_i \rightarrow \beta(a_i, d_i)\}_{i=1}^n} \underline{\mathrm{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n c_i, \beta(b, \otimes_{i=1}^n d_i)).$$

Hence, in order to define (6.17) it suffices to construct a compatible family of maps from $\mathbf{M}_n(a_1, \dots, a_n; b)$ to $\underline{\mathrm{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n c_i, \beta(b, \otimes_{i=1}^n d_i))$ indexed by $\{c_i \rightarrow \beta(a_i, d_i)\}_{i=1}^n$. These are given by the following composition

$$\begin{array}{c} \mathbf{M}_n(a_1, \dots, a_n; b) \\ \downarrow \\ \underline{\mathrm{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n \beta(a_i, d_i), \beta(b, \otimes_{i=1}^n d_i)) \\ \downarrow \\ \underline{\mathrm{Hom}}_{\mathcal{E}}(\otimes_{i=1}^n c_i, \beta(b, \otimes_{i=1}^n d_i)), \end{array}$$

where the first map is the component of (6.16) corresponding to the identity of $d := \otimes_{i=1}^n d_i$, and the second map is induced by the tensorisation $\otimes_{i=1}^n c_i \rightarrow \otimes_{i=1}^n \beta(a_i, d_i)$ of the elements of $\{c_i \rightarrow \beta(a_i, d_i)\}_{i=1}^n$. We obtain in this way a compatible family of maps and therefore a well-defined algebra $\bar{\beta}$. This construction is functorial in β and gives rise to the (quasi-)inverse to (6.12). \square

6.3. Elmendorf-Mandell's model structure. Recall from §4.1 that \mathcal{S} denotes the category of symmetric spectra endowed with Shipley's stable positive model structure. Given a sSet-enriched multicategory \mathbf{M} , Elmendorf-Mandell constructed in [16, Thm. 1.3] a simplicial model structure on the category $\mathbf{M}\text{-Alg}_{\mathrm{sSet}}(\mathcal{S}^{\otimes})$. The weak equivalences (resp. fibrations) are the objectwise weak equivalences (resp. fibrations) in \mathcal{S} . In this subsection we extend this result to a suitable category of presheaves; see Theorem 6.20 below.

Proposition 6.18. *Given a small symmetric monoidal category \mathcal{D} , the category of presheaves $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})$ carries a simplicial cofibrantly generated symmetric monoidal model structure. The weak equivalences (resp. fibrations) are the objectwise weak equivalences (resp. fibrations) in \mathcal{S} . The symmetric monoidal structure is given by the Day convolution product.*

Proof. Recall first that \mathcal{S} is a simplicial cofibrantly generated symmetric monoidal model category. In particular, it satisfies the pushout product axiom. Making use of [21, Thm. 11.6.1], one then obtains a simplicial cofibrantly generated projective model structure on $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})$. Given objects $d_1, d_2 \in \mathcal{D}$ and maps $i : s_1 \rightarrow s_2$ and $i' : s'_1 \rightarrow s'_2$ in \mathcal{S} , we have $(d_1 \otimes i) \square (d_2 \otimes i') \simeq (d_1 \otimes d_2) \otimes (i \square i')$, where $-\square-$ stands for the pushout-product. The generating (trivial) cofibrations of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})$ are of the form $d \otimes (i : s_1 \rightarrow s_2)$, with i a generating (trivial) cofibration of \mathcal{S} . Moreover, for every $d \in \mathcal{D}$ the functor $\mathcal{S} \rightarrow \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}), s \mapsto d \otimes s$, is a left Quillen

functor. Therefore, since \mathcal{S} satisfies the pushout product axiom, we conclude that $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ also satisfies the product pushout axiom. This achieves the proof. \square

Notation 6.19. Let \mathcal{S} be a set of morphisms in $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ for which the left Bousfield localization $L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ of $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ remains a simplicial cofibrantly generated symmetric monoidal model category.

Theorem 6.20. *The category $\mathbf{M}\text{-Alg}_{\text{ssSet}}(L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S}))^{\otimes}$ carries a simplicial cofibrantly generated model structure. The weak equivalences (resp. fibrations) are the objectwise weak equivalences (resp. fibrations) in $L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$.*

Proof. The proof is similar to the proof of [16, Thm. 1.3]. As explained in [16, pages 51-52], we have adjunctions

$$\begin{array}{ccc} \mathbf{M}\text{-Alg}_{\text{ssSet}}(L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S}))^{\otimes} & & (L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S}))^{\text{obj}(\mathbf{M})} \\ \mathbb{M}(-) \uparrow \downarrow \text{forget} & & \uparrow \downarrow \text{evaluation at } a \\ (L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S}))^{\text{obj}(\mathbf{M})} & & L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S}) \end{array}$$

with $a \in \text{obj}(\mathbf{M})$. Note that $(L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S}))^{\text{obj}(\mathbf{M})}$ inherits from $L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ a simplicial cofibrantly generated model structure. The general results [16, Props. 11.5, 11.6 and 11.8] hold *mutatis mutandis* in this broader setting. Hence, it remains only to show the analogue of [16, Lem. 11.7] which by a standard filtered colimit argument reduces to the following claim: *given an \mathbf{M} -algebra α in $L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ and a morphism $\eta : F \rightarrow H$ in $L_{\mathcal{S}}\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$, consider the following pushout square*

$$(6.21) \quad \begin{array}{ccc} \mathbb{M}(\iota_a(F)) & \xrightarrow{\quad} & \alpha \\ \mathbb{M}(\iota_a(\eta)) \downarrow & \lrcorner & \downarrow \\ \mathbb{M}(\iota_a(H)) & \longrightarrow & \mathbb{M}(\iota_a(H)) \amalg_{\mathbb{M}(\iota_a(F))} \alpha. \end{array}$$

If $\eta : F \rightarrow H$ is a generating trivial cofibration, then the right vertical map is a weak equivalence. As explained in [16, page 56], this map admits a factorization

$$\alpha = \alpha_0 \rightarrow \alpha_1 \rightarrow \cdots \rightarrow \text{colim}_k \alpha_k = \mathbb{M}(\iota_a(H)) \amalg_{\mathbb{M}(\iota_a(F))} \alpha,$$

where the objects α_k are determined by a pushout square

$$(6.22) \quad \begin{array}{ccc} \mathbb{U}_k \alpha \otimes_{\Sigma_k} Q_{k-1}^k & \xrightarrow{\quad} & \mathbb{U}_k \alpha \otimes_{\Sigma_k} \iota_a(H^{\otimes k}) \\ \downarrow & \lrcorner & \downarrow \\ \alpha_{k-1} & \longrightarrow & \alpha_k. \end{array}$$

Therefore, it suffices to show that the maps $\alpha_{k-1} \rightarrow \alpha_k$ are weak equivalences. As explained in *loc. cit.*, the quotient $\iota_a(H^{\otimes k})/Q_{k-1}^k$ is isomorphic to $\iota_a((H/F)^{\otimes k})$. Since by hypothesis η is a generating trivial cofibration, H/F is weakly equivalent to the initial=terminal object and hence $\mathbb{U}_k H \otimes_{\Sigma_k} \iota_a((H/F)^{\otimes k})$ is also weakly equivalent to the initial=terminal object. This implies that the upper horizontal map in (6.22) is a weak equivalence. Let us now show that (6.22) is an homotopy pushout square; which implies that the maps $\alpha_{k-1} \rightarrow \alpha_k$ are weak equivalences and hence finishes the proof. We can assume without loss of generality that η is a generating cofibration. Moreover, since homotopy colimits are computed objectwise, we can

replace $L_S \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ by $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$. Recall from the proof of Proposition 6.18 that the generating cofibrations of $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{C})$ are of the form $d \otimes (i : s_1 \rightarrow s_2)$, with $d \in \mathcal{D}$ and i a generating cofibration of \mathcal{S} . Making use of Theorem 6.11 (with $\mathcal{E} = \text{sSet}$ and $\mathcal{C} = \mathcal{S}$), i.e. of the equivalence

$$\mathbf{M}\text{-Alg}_{\text{sSet}}(\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes}) \simeq (\mathbf{M} \otimes_{\text{sSet}} (\mathcal{D}^{\text{op}})^{\otimes})\text{-Alg}_{\text{sSet}}(\mathcal{S}^{\otimes}),$$

it suffices then only to show that the diagram analogue to (6.22) (with ι_a replaced by $\iota_{(a,d)}$ and H by s_2) is an homotopy pushout square. This follows from [16, Lems. 11.7 and 12.6] and so the proof is finished. \square

6.4. Quillen equivalences. Note that every multicategory \mathbf{M} has an underlying category $C_{\mathbf{M}}$. The objects are those of \mathbf{M} and $\text{Hom}_{C_{\mathbf{M}}}(a, b) := \mathbf{M}_1(a; b)$. When \mathbf{M} is a \mathcal{E} -enriched multicategory, the category $C_{\mathbf{M}}$ is enriched over \mathcal{E} . Every (\mathcal{E} -enriched) multifunctor $h : \mathbf{M} \rightarrow \mathbf{M}'$ gives rise to a (\mathcal{E} -enriched) functor $C_h : C_{\mathbf{M}} \rightarrow C_{\mathbf{M}'}$. Recall from [16, Def. 12.2] that a sSet -enriched multifunctor $h : \mathbf{M} \rightarrow \mathbf{M}'$ is a *weak equivalence* if the following two conditions hold:

- (i) the maps $\mathbf{M}_n(a_1, \dots, a_n; b) \rightarrow \mathbf{M}'_n(h(a_1), \dots, h(a_n); h(b))$ are weak equivalences of simplicial sets and;
- (ii) the functor $\pi_0(C_f) : \pi_0(C_{\mathbf{M}}) \xrightarrow{\sim} \pi_0(C_{\mathbf{M}'})$ is an equivalence of categories.

Elmendorf-Mandell proved in [16, Thm. 1.4] that a weak equivalence between sSet -enriched multicategories gives rise to a Quillen equivalence between the associated categories of algebras. We generalize this result as follows:

Theorem 6.23. *Given a weak equivalence $h : \mathbf{M} \rightarrow \mathbf{M}'$ between sSet -enriched multicategories, one has an induced Quillen equivalence*

$$(6.24) \quad \begin{array}{c} \mathbf{M}'\text{-Alg}_{\text{sSet}}(L_S \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes}) \\ \uparrow h_* \quad \downarrow h^* \\ \mathbf{M}\text{-Alg}_{\text{sSet}}(L_S \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes}). \end{array}$$

Proof. By construction, the right adjoint functor h^* preserves and reflects weak equivalences and fibrations. Hence, it suffices to show that for every cofibrant \mathbf{M} -algebra α in $L_S \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ the counit $\alpha \rightarrow h^* h_*(\alpha)$ is a weak equivalence. Thanks to Theorem 6.11 (with $\mathcal{E} = \text{sSet}$ and $\mathcal{C} = \mathcal{S}$), we have the following equivalences

$$\begin{aligned} \mathbf{M}\text{-Alg}_{\text{sSet}}(\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes}) &\simeq (\mathbf{M} \otimes_{\text{sSet}} (\mathcal{D}^{\text{op}})^{\otimes})\text{-Alg}_{\text{sSet}}(\mathcal{S}^{\otimes}) \\ \mathbf{M}'\text{-Alg}_{\text{sSet}}(\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes}) &\simeq (\mathbf{M}' \otimes_{\text{sSet}} (\mathcal{D}^{\text{op}})^{\otimes})\text{-Alg}_{\text{sSet}}(\mathcal{S}^{\otimes}). \end{aligned}$$

Moreover, since by assumption h is a weak equivalence, the sSet -enriched multifunctor $h \otimes_{\text{sSet}} \text{id} : \mathbf{M} \otimes_{\text{sSet}} (\mathcal{D}^{\text{op}})^{\otimes} \rightarrow \mathbf{M}' \otimes_{\text{sSet}} (\mathcal{D}^{\text{op}})^{\otimes}$ is also a weak equivalence. Hence, by applying [16, Thm. 1.4] to $h \otimes_{\text{sSet}} \text{id}$ one obtains the Quillen equivalence

$$(6.25) \quad \begin{array}{c} \mathbf{M}'\text{-Alg}_{\text{sSet}}(\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes}) \\ \uparrow h_* \quad \downarrow h^* \\ \mathbf{M}\text{-Alg}_{\text{sSet}}(\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes}). \end{array}$$

Now, note that the functors h_* and h^* of adjunctions (6.24)-(6.25) are exactly the same. Moreover, every cofibrant \mathbf{M} -algebra α in $L_S \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$ is also a cofibrant \mathbf{M} -algebra α in $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$. Hence, we conclude that $\alpha \rightarrow h^* h_*(\alpha)$ is a weak equivalence in $\mathbf{M}\text{-Alg}_{\text{sSet}}(L_S \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})^{\otimes})$. This achieves the proof. \square

7. COMMUTATIVE MONOIDS IN SYMMETRIC SPECTRA

Given a left proper cellular symmetric monoidal model category $(\mathcal{C}, \otimes, \mathbf{1})$ and a cofibrant object $c \in \mathcal{C}$, Hovey constructed in [22] the symmetric monoidal model category $\mathrm{Sp}_c^\Sigma(\mathcal{C})$ of c -symmetric spectra on \mathcal{C} .

Proposition 7.1. *Let d be a (unital) commutative monoid in \mathcal{C} . Given a morphism $m : c \rightarrow d$ in \mathcal{C} , one obtains to a (unital) commutative monoid $\mathbf{d} \in \mathrm{Sp}_c^\Sigma(\mathcal{C})$ with $\mathbf{d}_n = d, n \geq 0$, and with structure morphisms $c \otimes d \xrightarrow{m \otimes \mathrm{id}} d \otimes d \xrightarrow{\mathrm{mult}} d$.*

Proof. Recall from [22, §7] the construction of the symmetric monoidal category \mathcal{C}^Σ of symmetric sequences. Given symmetric sequences a, b, c , one has

$$(7.2) \quad \mathrm{Hom}_{\mathcal{C}^\Sigma}(a \otimes b, c) \simeq \prod_{p,q} \mathrm{Hom}_{\mathcal{C}^{\Sigma_p \times \Sigma_q}}(a_p \otimes b_q, c_{p+q}).$$

Now, consider the symmetric sequence $(0, c, \dots, 0, \dots)$ (where 0 stands for the initial object) and the associated free commutative monoid $\mathrm{Sym}(c) := (\mathbf{1}, c, \dots, c^{\otimes n}, \dots)$. As explained in *loc. cit.*, $\mathrm{Sp}_c^\Sigma(\mathcal{C})$ identifies with the category of left modules over $\mathrm{Sym}(c)$. Hence, a (unital) commutative monoid \mathbf{d} in $\mathrm{Sp}_c^\Sigma(\mathcal{C})$ is the same data as a (unital) commutative monoid $\mathbf{d} = (d_0, d_1, \dots, d_n, \dots)$ in \mathcal{C}^Σ endowed with a morphism $m : c \rightarrow d_1$ in \mathcal{C} . Let us then take $d_n := d$ for all $n \geq 0$ endowed with the trivial Σ_n -action. The multiplication of c' , combined with isomorphism (7.2), allow us to conclude that these choices give rise to a well-defined (unital) commutative monoid $\mathbf{d} \in \mathcal{C}^\Sigma$. Making use of the morphism $m : c \rightarrow d$ one then obtains the desired (unital) commutative monoid $\mathbf{d} \in \mathrm{Sp}_c^\Sigma(\mathcal{C})$. \square

8. A KEY COMMUTATIVE MONOID

Recall from the proof of [55, Theorem 2.1] that the derivator $\mathcal{M}\mathrm{ot}_{A^1}(k)$ admits a symmetric monoidal Quillen model $L_S\mathrm{Fun}(\mathrm{dgc}\mathrm{at}_f(k)^{\mathrm{op}}, \mathrm{Sp}^\Sigma)$. Some explanations are in order: $\mathrm{dgc}\mathrm{at}_f(k)$ is a small symmetric monoidal category, the symmetric monoidal structure is given by the Day convolution product, and the set S (denoted by $\{S, \mathrm{loc}\}$ in *loc. cit.*) implements the A^1 -homotopy and localization properties.

Notation 8.1 (Quillen model). Let $M^Q := L_S\mathrm{Fun}(\mathrm{dgc}\mathrm{at}_f(k)^{\mathrm{op}}, S)$ be the symmetric monoidal Quillen model of $\mathcal{M}\mathrm{ot}_{A^1}(k)$. Note that the Quillen equivalence $\mathrm{Sp}^\Sigma \simeq S$ gives rise to a Quillen equivalence $L_S\mathrm{Fun}(\mathrm{dgc}\mathrm{at}_f(k)^{\mathrm{op}}, \mathrm{Sp}^\Sigma) \simeq M^Q$. Note also that M^Q is naturally enriched over S . Following [55, Rk. 7.6], the universal A^1 -localizing invariant \mathcal{U}_{A^1} is induced by the symmetric monoidal functor

$$\mathrm{dgc}\mathrm{at}(k) \longrightarrow M^Q \quad \mathcal{A} \mapsto (\mathcal{B} \mapsto \Sigma^\infty(Nw\mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathcal{A})_+))$$

where $w\mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathcal{A})$ stands for the category of quasi-isomorphisms of $\mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathcal{A})$, $Nw\mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathcal{A})$ for its nerve, and $\Sigma^\infty(-)_+$ for the suspension symmetric spectrum.

The category $\mathrm{Fun}(\mathrm{Sm}(k)^{\mathrm{op}}, M^Q)$, endowed with the projective model structure and the objectwise tensor product, is a symmetric monoidal model category. Given a k -scheme $X \in \mathrm{Sm}(k)$, we will still denote by X the associated constant presheaf. Following Ayoub [1, §4.4-4.5], we can then consider the symmetric monoidal model category of (\mathbb{P}^1, ∞) -spectra (with coefficients in M^Q)

$$\mathrm{SH}(k; M^Q) := \mathrm{Sp}_{(\mathbb{P}^1, \infty)}^\Sigma(L_{A^1, \mathrm{Nis}} \mathrm{Fun}(\mathrm{Sm}(k)^{\mathrm{op}}, M^Q)).$$

By construction, we have a Quillen adjunction

$$(8.2) \quad \begin{array}{ccc} & \mathrm{SH}(k; \mathbf{M}^Q) & \\ & \uparrow \Sigma^\infty(-+) \quad \downarrow (-)_0 & \\ L_{A^1, \mathrm{Nis}} \mathrm{Fun}(\mathrm{Sm}(k)^{\mathrm{op}}, \mathbf{M}^Q), & & \end{array}$$

where $(-)_0$ stands for the 0^{th} component functor. Note that since the inclusion $\mathrm{Sm}'(k) \hookrightarrow \mathrm{Sm}(k)$ is an equivalence of Nisnevich sites (see §4.3), one can (and will) replace $\mathrm{Sm}(k)$ by $\mathrm{Sm}'(k)$. Note also that $\mathrm{SH}(k; \mathbf{M}^Q)$ is naturally enriched over \mathbf{M}^Q , and consequently over \mathcal{S} . In this section, we construct a commutative monoid in $\mathrm{SH}(k; \mathbf{M}^Q)$. This is divided into two steps:

- (i) **First step:** making use of the general theory of algebras over multicategories developed in §6, we start by constructing a commutative monoid $\mathrm{KGL}_{\mathrm{nc}}$ in $\mathrm{Fun}(\mathrm{Sm}'(k)^{\mathrm{op}}, \mathbf{M}^Q)$; see Theorem 8.20.
- (ii) **Second step:** making use of the general theory of commutative monoids in symmetric spectra developed in §7, we then promote $\mathrm{KGL}_{\mathrm{nc}}$ to a commutative monoid $\mathrm{KGL}_{\mathrm{nc}}$ in $\mathrm{SH}(k; \mathbf{M}^Q)$; see Theorem 8.25.

The commutative monoid $\mathrm{KGL}_{\mathrm{nc}}$ will play a key role in the proof of Theorem 2.1.

First step.

Notation 8.3. Given a dg category \mathcal{A} and a k -scheme $X \in \mathrm{Sm}'(k)$, let $\mathrm{perf}_{\mathrm{dg}}^{\mathrm{flat}}(X)$ be the dg subcategory of $\mathrm{perf}_{\mathrm{dg}}(X)$ of flat perfect complexes, $C(\mathcal{A}, X)$ the category of contravariant dg functors from \mathcal{A} to $\mathrm{perf}_{\mathrm{dg}}^{\mathrm{flat}}(X)$, and $w(\mathcal{A}, X)$ the category of quasi-isomorphisms of $C(\mathcal{A}, X)$. The categories $C(\mathcal{A}, X)$ and $w(\mathcal{A}, X)$ carry an objectwise symmetric monoidal structure.

Lemma 8.4. (i) *Every morphism $X \rightarrow Y$ in $\mathrm{Sm}'(k)$ gives rise to a symmetric monoidal functor $C(\mathcal{A}, Y) \rightarrow C(\mathcal{A}, X)$ which restricts to $w(\mathcal{A}, Y) \rightarrow w(\mathcal{A}, X)$;*
(ii) *Every dg functor $\mathcal{A} \rightarrow \mathcal{B}$ gives rise to a symmetric monoidal functor $C(\mathcal{B}, X) \rightarrow C(\mathcal{A}, X)$ which restricts to $w(\mathcal{B}, X) \rightarrow w(\mathcal{A}, X)$.*

Proof. The inverse image dg functor $\mathrm{perf}_{\mathrm{dg}}(Y) \rightarrow \mathrm{perf}_{\mathrm{dg}}(X)$ is symmetric monoidal, preserves flat perfect complexes, and also quasi-isomorphisms between them. This implies item (i). Item (ii) is clear. \square

Let us now restrict ourselves to the subcategory $\mathrm{dgc}_{\mathrm{f}}(k) \subset \mathrm{dgc}_{\mathrm{at}}(k)$. Thanks to Lemma 8.4 one has a fibered category, which by a standard procedure can be strictified into a genuine presheaf of categories

$$(8.5) \quad w := w(-, -) : (\mathrm{dgc}_{\mathrm{f}}(k) \times \mathrm{Sm}'(k))^{\mathrm{op}} \longrightarrow \mathrm{CAT}.$$

Recall from Example 6.2 the definition of the CAT-enriched operad $\mathcal{G}\Sigma$.

Proposition 8.6. *The presheaf (8.5) carries an action of $\mathcal{G}\Sigma$, i.e. it belongs to the category $\mathcal{G}\Sigma\text{-Alg}_{\mathrm{CAT}}(\mathrm{Fun}((\mathrm{dgc}_{\mathrm{f}}(k) \times \mathrm{Sm}'(k))^{\mathrm{op}}, \mathrm{CAT}))^{\otimes}$.*

Proof. One needs to construct functors $\mathcal{G}\Sigma_n \rightarrow \underline{\mathrm{Hom}}_{\mathrm{CAT}}(\star_{i=1}^n w, w)$, $n \geq 0$, preserving the Σ_n -action, the “unit” element, and the composition “multiproduct”. Recall from Lemma 6.7 (with $\mathcal{E} = \mathcal{C} = \mathrm{CAT}$ and $\mathcal{D} = \mathrm{dgc}_{\mathrm{f}}(k) \times \mathrm{Sm}'(k)$) that

$$\underline{\mathrm{Hom}}_{\mathrm{CAT}}(\star_{i=1}^n w, w) \simeq \lim_{\{c_i \rightarrow w(\mathcal{A}_i, X_i)\}_{i=1}^n} \underline{\mathrm{Hom}}_{\mathrm{CAT}}(\otimes_{i=1}^n c_i, w(\otimes_{i=1}^n \mathcal{A}_i, X_i)).$$

Therefore, it suffices to construct a compatible family of functors

$$(8.7) \quad \mathcal{G}\Sigma_n \longrightarrow \underline{\text{Hom}}_{\text{CAT}}(\otimes_{i=1}^n c_i, w(\otimes_{i=1}^n (\mathcal{A}_i, X_i)))$$

indexed by $\{c_i \rightarrow w(\mathcal{A}_i, X_i)\}_{i=1}^n$. Let us start by constructing a functor

$$(8.8) \quad \mathcal{G}\Sigma_n \longrightarrow \underline{\text{Hom}}_{\text{CAT}}(\otimes_{i=1}^n w(\mathcal{A}_i, X_i), w(\otimes_{i=1}^n (\mathcal{A}_i, X_i))).$$

For each $\sigma \in \Sigma_n$, consider the dg functor

$$(8.9) \quad \begin{aligned} \otimes_{i=1}^n \text{perf}_{\text{dg}}^{\text{flat}}(X_i) &\longrightarrow \text{perf}_{\text{dg}}^{\text{flat}}(X_1 \times \cdots \times X_n) \\ \{\mathcal{F}_i\}_{i=1}^n &\mapsto \otimes_{i=1}^n \pi_{\sigma(i)}^*(\mathcal{F}_{\sigma(i)}), \end{aligned}$$

where π_i stands for the projection map $\prod_{i=1}^n X_i \rightarrow X_i$. Making use of it, one constructs the functor $\otimes_{i=1}^n C(\mathcal{A}_i, X_i) \rightarrow C(\otimes_{i=1}^n \mathcal{A}_i, \prod_{i=1}^n X_i)$ that sends the family $\{F_i : \mathcal{A}_i^{\text{op}} \rightarrow \text{perf}_{\text{dg}}^{\text{flat}}(X_i)\}_{i=1}^n$ to the composition

$$(\otimes_{i=1}^n \mathcal{A}_i)^{\text{op}} \xrightarrow{\otimes_{i=1}^n F_i} \otimes_{i=1}^n \text{perf}_{\text{dg}}^{\text{flat}}(X_i) \xrightarrow{(8.9)} \text{perf}_{\text{dg}}^{\text{flat}}(X_1 \times \cdots \times X_n).$$

Note that this latter functor restricts to the subcategories of quasi-isomorphisms

$$T_\sigma : \otimes_{i=1}^n w(\mathcal{A}_i, X_i) \longrightarrow w(\otimes_{i=1}^n \mathcal{A}_i, X_1 \times \cdots \times X_n) = w(\otimes_{i=1}^n (\mathcal{A}_i, X_i)).$$

The above functor (8.8) is then defined by sending $\sigma \in \Sigma_n$ to T_σ . Given any other $\tau \in \Sigma_n$, the unique isomorphism $\sigma \xrightarrow{\sim} \tau$ in the groupoid $\mathcal{G}\Sigma_n$ is mapped to the natural transformation $T_\sigma \xrightarrow{\sim} T_\tau$ induced by the coherence isomorphism $\otimes_{i=1}^n \pi_{\sigma(i)}^*(\mathcal{F}_{\sigma(i)}) \simeq \otimes_{i=1}^n \pi_{\tau(i)}^*(\mathcal{F}_{\tau(i)})$. Finally, the searched compatible family of functors (8.7) is defined by composing (8.8) with the functor

$$\underline{\text{Hom}}_{\text{CAT}}(\otimes_{i=1}^n w(\mathcal{A}_i, X_i), w(\otimes_{i=1}^n (\mathcal{A}_i, X_i))) \longrightarrow \underline{\text{Hom}}_{\text{CAT}}(\otimes_{i=1}^n c_i, w(\otimes_{i=1}^n (\mathcal{A}_i, X_i)))$$

induced by the tensorization $\otimes_{i=1}^n c_i \rightarrow \otimes_{i=1}^n w(\mathcal{A}_i, X_i)$ of the elements of $\{c_i \rightarrow w(\mathcal{A}_i, X_i)\}_{i=1}^n$. This achieves the proof. \square

Now, consider the following presheaf of simplicial sets

$$(8.10) \quad (\text{dgcat}_f(k) \times \text{Sm}'(k))^{\text{op}} \xrightarrow{w} \text{CAT} \xrightarrow{N} \text{sSet},$$

where N stands for the nerve functor. By applying the symmetric monoidal functor $N : \text{CAT} \rightarrow \text{sSet}$ to $\mathcal{G}\Sigma$ one obtains a sSet -enriched operad $N(\mathcal{G}\Sigma)$.

Proposition 8.11. *The presheaf (8.10) carries an action of $N(\mathcal{G}\Sigma)$, i.e. it belongs to the category $N(\mathcal{G}\Sigma)\text{-Alg}_{\text{sSet}}(\text{Fun}((\text{dgcat}_f(k) \times \text{Sm}'(k))^{\text{op}}, \text{sSet})^{\otimes})$.*

Proof. Consider the induced functor

$$N_* : \text{Fun}((\text{dgcat}_f(k) \times \text{Sm}'(k))^{\text{op}}, \text{CAT}) \longrightarrow \text{Fun}((\text{dgcat}_f(k) \times \text{Sm}'(k))^{\text{op}}, \text{sSet}).$$

Thanks to Lemma 8.14 below, N_* is symmetric monoidal. One needs then to construct maps of simplicial maps

$$(8.12) \quad N(\mathcal{G}\Sigma_n) \longrightarrow \underline{\text{Hom}}_{\text{sSet}}(\star_{i=1}^n N_*(w), N_*(w)) \quad n \geq 0$$

preserving the Σ_n -action, the “unit” element, and the composition “multiproduct”. As explained in the proof of Proposition 8.6, the $\mathcal{G}\Sigma$ -algebra structure of the presheaf (8.5) consists of a sequence of functors

$$(8.13) \quad \mathcal{G}\Sigma_n \rightarrow \underline{\text{Hom}}_{\text{CAT}}(\star_{i=1}^n w, w) \quad n \geq 0.$$

Making use of them, one then defines (8.12) to be the following composition

$$N(\mathcal{G}\Sigma_n) \xrightarrow{N((8.13))} N(\underline{\text{Hom}}_{\text{CAT}}(\star_{i=1}^n w, w)) \xrightarrow{N_*} \underline{\text{Hom}}_{\text{sSet}}(\star_{i=1}^n N_*(w), N_*(w)).$$

This achieves the proof. \square

Lemma 8.14. *The above functor N_* is symmetric monoidal.*

Proof. Let $u \in \text{Fun}((\text{dgc}at_f(k) \times \text{Sm}'(k))^{\text{op}}, \text{CAT})$. Recall from the proof of Lemma 6.7 that we have the canonical presentation $\text{colim}_{(\mathcal{A}, X) \otimes c \rightarrow u} (\mathcal{A}, X) \otimes c \simeq u$. By applying it to the functor N_* , we obtain an induced map

$$(8.15) \quad \text{colim}_{(\mathcal{A}, X) \otimes c \rightarrow u} N_*((\mathcal{A}, X) \otimes c) \longrightarrow N_*(u).$$

We claim that (8.15) is an isomorphism. On one hand, since the functor N is fully-faithful and preserves coproducts, the left-hand-side of (8.15) identifies with $\text{colim}_{(\mathcal{A}, X) \otimes N(c) \rightarrow N_*(u)} (\mathcal{A}, X) \otimes N(c)$. On the other hand, we have the canonical presentation $\text{colim}_{(\mathcal{A}, X) \otimes K \rightarrow N_*(u)} (\mathcal{A}, X) \otimes K \simeq N_*(u)$ with $K \in \text{sSet}$. Using the fact that the diagram $\{(\mathcal{A}, X) \otimes N(c) \rightarrow N_*(u)\}$ is cofinal in $\{(\mathcal{A}, X) \otimes K \rightarrow N_*(u)\}$, we then conclude that (8.15) is an isomorphism. The proof follows now from the combination of isomorphism (8.15) with the properties of the Day convolution product (see §6.2). \square

Now, consider the following presheaf of symmetric spectra

$$(8.16) \quad (\text{dgc}at_f(k) \times \text{Sm}'(k))^{\text{op}} \xrightarrow{w} \text{CAT} \xrightarrow{N} \text{sSet} \xrightarrow{\Sigma^\infty(-)_+} \mathcal{S}.$$

Proposition 8.17. *The presheaf (8.16) carries an action of $N(\mathcal{G}\Sigma)$, i.e. it belongs to the category $N(\mathcal{G}\Sigma)\text{-Alg}_{\text{sSet}}(\text{Fun}((\text{dgc}at_f(k) \times \text{Sm}'(k))^{\text{op}}, \mathcal{S})^\otimes)$.*

Proof. The category \mathcal{S} of symmetric spectra is enriched over sSet and the colimit preserving symmetric monoidal functor $\Sigma^\infty(-)_+ : \text{sSet} \rightarrow \mathcal{S}$ preserves this enrichment. Hence, thanks to Remark 6.6, we obtain a sSet -enriched multifunctor

$$\text{Fun}((\text{dgc}at_f(k) \times \text{Sm}'(k))^{\text{op}}, \text{sSet})^\otimes \longrightarrow \text{Fun}((\text{dgc}at_f(k) \times \text{Sm}'(k))^{\text{op}}, \mathcal{S})^\otimes.$$

By pre-composing it with the $N(\mathcal{G}\Sigma)$ -algebra structure of (8.10), one then obtains an action of $N(\mathcal{G}\Sigma)$ on (8.16). This achieves the proof. \square

Remark 8.18. The symmetric monoidal categories $\text{Fun}(\text{dgc}at_f(k)^{\text{op}}, \mathcal{S})$ and M^Q are the same. Hence, the classical symmetric monoidal equivalence

$$\text{Fun}((\text{dgc}at_f(k) \times \text{Sm}'(k))^{\text{op}}, \mathcal{S}) \simeq \text{Fun}(\text{Sm}'(k)^{\text{op}}, \text{Fun}(\text{dgc}at_f(k)^{\text{op}}, \mathcal{S}))$$

gives rise to an equivalence

$$(8.19) \quad \begin{aligned} & N(\mathcal{G}\Sigma)\text{-Alg}_{\text{sSet}}(\text{Fun}((\text{dgc}at_f(k) \times \text{Sm}'(k))^{\text{op}}, \mathcal{S})^\otimes) \\ & \simeq N(\mathcal{G}\Sigma)\text{-Alg}_{\text{sSet}}(\text{Fun}(\text{Sm}'(k)^{\text{op}}, \text{M}^Q)^\otimes). \end{aligned}$$

Consequently, the above presheaf (8.16) belongs also to (8.19)

Theorem 8.20. *There exists a fibrant model KGL_{nc} of (8.16) in the symmetric monoidal model category $\text{Fun}(\text{Sm}'(k)^{\text{op}}, \text{M}^Q)$ which is a commutative monoid.*

Proof. Note that the simplicial sets $N(\mathcal{G}\Sigma_n)$, $n \geq 0$, are contractible. Hence, the sSet -enriched projection multifunctor $N(\mathcal{G}\Sigma) \rightarrow \text{Comm}$ (see Example 6.2) is a weak equivalence in the sense of §6.4. The same holds for the sSet -enriched multifunctor $h : N(\mathcal{G}\Sigma) \otimes_{\text{sSet}} (\text{Sm}'(k)^{\text{op}})^\otimes \rightarrow \text{Comm} \otimes_{\text{sSet}} (\text{Sm}'(k)^{\text{op}})^\otimes$, where $(\text{Sm}'(k)^{\text{op}})^\otimes$ is

enriched over \mathbf{sSet} in the trivial way. Hence, thanks to Theorems 6.20 and 6.23 (with $\mathcal{D} = \mathbf{dgc}_{\mathbf{f}}(k)$) one obtains a Quillen equivalence

$$(8.21) \quad \begin{array}{c} \mathbf{Comm} \otimes_{\mathbf{sSet}} (\mathbf{Sm}'(k)^{\mathrm{op}})^{\otimes} \text{-} \mathbf{Alg}_{\mathbf{sSet}}((\mathbf{M}^Q)^{\otimes}) \\ \uparrow h_* \quad \downarrow h^* \\ N(\mathcal{G}\Sigma) \otimes_{\mathbf{sSet}} (\mathbf{Sm}'(k)^{\mathrm{op}})^{\otimes} \text{-} \mathbf{Alg}_{\mathbf{sSet}}((\mathbf{M}^Q)^{\otimes}). \end{array}$$

Moreover, thanks to Theorem 6.11 (with $\mathcal{C} = \mathbf{M}^Q$ and $\mathcal{D} = \mathbf{Sm}'(k)$) we have

$$\begin{aligned} \mathbf{Comm} \otimes_{\mathbf{sSet}} (\mathbf{Sm}'(k)^{\mathrm{op}})^{\otimes} \text{-} \mathbf{Alg}_{\mathbf{sSet}}((\mathbf{M}^Q)^{\otimes}) &\simeq \mathbf{Comm}\text{-}\mathbf{Alg}_{\mathbf{sSet}}(\mathbf{Fun}(\mathbf{Sm}'(k)^{\mathrm{op}}, \mathbf{M}^Q)^{\otimes}) \\ N(\mathcal{G}\Sigma) \otimes_{\mathbf{sSet}} (\mathbf{Sm}'(k)^{\mathrm{op}})^{\otimes} \text{-} \mathbf{Alg}_{\mathbf{sSet}}((\mathbf{M}^Q)^{\otimes}) &\simeq N(\mathcal{G}\Sigma)\text{-}\mathbf{Alg}_{\mathbf{sSet}}(\mathbf{Fun}(\mathbf{Sm}'(k)^{\mathrm{op}}, \mathbf{M}^Q)^{\otimes}). \end{aligned}$$

Note that the right-hand-side of the first equivalence is the category of commutative monoids in $\mathbf{Fun}(\mathbf{Sm}'(k)^{\mathrm{op}}, \mathbf{M}^Q)$. Hence, the proof follows now from Quillen equivalence (8.21) combined with the above Remark 8.18. \square

Lemma 8.22. *The evaluation of the presheaf (8.16) at a k -scheme $X \in \mathbf{Sm}'(k)$ identifies with $U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X))$.*

Proof. Thanks to the definition of (8.16) and Notation 8.1, one needs to show that the following presheaves of symmetric spectra on $\mathbf{dgc}_{\mathbf{f}}(k)$ are isomorphic

$$\mathcal{B} \mapsto \Sigma^{\infty}(Nw(\mathcal{B}, X)_{+}) \quad \mathcal{B} \mapsto \Sigma^{\infty}(Nw\mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathrm{perf}_{\mathrm{dg}}(X))_{+}).$$

On one hand, $\mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathrm{perf}_{\mathrm{dg}}(X))$ identifies with the category of contravariant dg functors from \mathcal{B} to $\mathrm{perf}_{\mathrm{dg}}(X)$. On the other hand, $C(\mathcal{B}, X)$ is by definition the category of contravariant dg functors from \mathcal{B} to $\mathrm{perf}_{\mathrm{dg}}^{\mathrm{flat}}(X)$. Since the natural inclusion $\mathrm{perf}_{\mathrm{dg}}^{\mathrm{flat}}(X) \hookrightarrow \mathrm{perf}_{\mathrm{dg}}(X)$ is a Morita equivalence and every object of $\mathrm{perf}_{\mathrm{dg}}(X)$ admits a functorial flat resolution, we conclude that the categories of quasi-isomorphisms $w(\mathcal{B}, X)$ and $w\mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathrm{perf}_{\mathrm{dg}}(X))$ are naturally equivalent. This implies our claim and consequently achieves the proof. \square

Proposition 8.23. *The commutative monoid KGL_{nc} of Theorem 8.20 is fibrant in the symmetric monoidal model category $L_{A^1, \mathrm{Nis}} \mathbf{Fun}(\mathbf{Sm}'(k)^{\mathrm{op}}, \mathbf{M}^Q)$.*

Proof. By construction, KGL_{nc} is fibrant in $\mathbf{Fun}(\mathbf{Sm}(k)^{\mathrm{op}}, \mathbf{M}^Q)$. It remains then only to show A^1 -homotopy invariance and Nisnevich descent. Thanks to Lemma 8.22, this is equivalent to the claim that the composed functor

$$\mathbf{Sm}'(k)^{\mathrm{op}} \xrightarrow{\mathrm{perf}_{\mathrm{dg}}(-)} \mathbf{Ho}(\mathbf{dgc}_{\mathbf{f}}(k)) \xrightarrow{U_{A^1}} \mathbf{Ho}(\mathbf{M}^Q)$$

is A^1 -homotopy invariant and satisfies Nisnevich descent. The latter claim follows from the fact that U_{A^1} is a localizing invariant; see [49, Thm. 3.1]. The former claim follows from Remark 5.3 (applied to $Y = \mathrm{Spec}(k[t])$) and from the fact that U_{A^1} is symmetric monoidal and A^1 -homotopy invariant. \square

Second step. Since \mathbf{M}^Q is enriched over \mathcal{S} , one has the Quillen adjunction

$$\begin{array}{c} \mathbf{M}^Q \\ \uparrow \downarrow \Gamma := \underline{\mathrm{Hom}}_{\mathcal{S}}(U_{A^1}(k), -) \\ \mathcal{S}, \end{array}$$

where ι is the unique homotopy colimit preserving symmetric monoidal functor sending the sphere spectrum \mathbb{S} to the \otimes -unit $U_{A^1}(k)$. By functoriality, one obtains the following Quillen adjunction

$$(8.24) \quad \begin{array}{c} L_{A^1, \text{Nis}} \text{Fun}(\text{Sm}'(k)^{\text{op}}, M^Q) \\ \uparrow \downarrow \Gamma \\ L_{A^1, \text{Nis}} \text{Fun}(\text{Sm}'(k)^{\text{op}}, \mathcal{S}). \end{array}$$

Theorem 8.25. *The commutative monoid KGL_{nc} of Theorem 8.20 gives rise to a commutative monoid KGL_{nc} in the symmetric monoidal model category $\text{SH}(k; M^Q)$.*

Proof. Thanks to Theorem 8.20 and the general Proposition 7.1 (with $c' := KGL_{\text{nc}}$), it suffices to construct a morphism $(\mathbb{P}^1, \infty) \rightarrow KGL_{\text{nc}}$ in $L_{A^1, \text{Nis}} \text{Fun}(\text{Sm}'(k)^{\text{op}}, M^Q)$. Moreover, since the functor ι in the above adjunction (8.24) sends (\mathbb{P}^1, ∞) to (\mathbb{P}^1, ∞) , it is enough to construct a morphism $m : (\mathbb{P}^1, \infty) \rightarrow \Gamma(KGL_{\text{nc}})$ in $L_{A^1, \text{Nis}} \text{Fun}(\text{Sm}'(k)^{\text{op}}, \mathcal{S})$. Thanks to Lemma 8.26 below, $\Gamma(KGL_{\text{nc}})$ identifies with homotopy algebraic K -theory KH . Since (\mathbb{P}^1, ∞) is cofibrant and $\Gamma(KGL_{\text{nc}})$ is fibrant, the morphisms in the homotopy category $\text{Ho}(L_{A^1, \text{Nis}} \text{Fun}(\text{Sm}'(k)^{\text{op}}, \mathcal{S}))$ from (\mathbb{P}^1, ∞) to $\Gamma(KGL_{\text{nc}})$ are in bijection with the elements of $KH_0(\mathbb{P}^1)$ which are sent to zero by the induced homomorphism

$$KH_0(\infty) : KH_0(\mathbb{P}^1) \longrightarrow KH_0(\text{Spec}(k)).$$

A canonical choice is the element $[\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\mathbb{P}^1}(-1)] \in KH_0(\mathbb{P}^1)$. By further choosing a representative of the induced morphism in the homotopy category, one then obtains the desired morphism $m : (\mathbb{P}^1, \infty) \rightarrow \Gamma(KGL_{\text{nc}})$. This achieves the proof. \square

Lemma 8.26. *The presheaf $\Gamma(KGL_{\text{nc}})$ identifies with homotopy algebraic K -theory $KH : \text{Sm}'(k)^{\text{op}} \rightarrow \mathcal{S}, X \mapsto KH(X)$.*

Proof. Thanks to Lemma 8.22, the evaluation of KGL_{nc} at a k -scheme $X \in \text{Sm}'(k)$ identifies with $U_{A^1}(\text{perf}_{\text{dg}}(X))$. Hence, by combining the definition of Γ with isomorphism (5.2), one concludes that the evaluation of $\Gamma(KGL_{\text{nc}})$ at X is isomorphic to $\underline{\text{Hom}}_{\mathcal{S}}(U_{A^1}(k), U_{A^1}(\text{perf}_{\text{dg}}(X))) \simeq KH(X)$. This achieves the proof. \square

Proposition 8.27. *The commutative monoid KGL_{nc} of Theorem 8.25 is fibrant in the symmetric monoidal model category $\text{SH}(k; M^Q)$.*

Proof. By definition of $\text{SH}(k; M^Q)$, one needs to show that KGL_{nc} is fibrant and that the canonical map $KGL_{\text{nc}} \rightarrow \underline{\text{Hom}}((\mathbb{P}^1, \infty), KGL_{\text{nc}})$ is an isomorphism. The first claim follows from Proposition 8.23. In what concerns the second claim

$$(8.28) \quad \underline{\text{Hom}}((\mathbb{P}^1, \infty), KGL_{\text{nc}}) \simeq \text{hofiber}(\underline{\text{Hom}}(\mathbb{P}^1, KGL_{\text{nc}}) \rightarrow KGL_{\text{nc}}).$$

Thanks to Lemma 8.29 below, $\underline{\text{Hom}}(\mathbb{P}^1, KGL_{\text{nc}}) \simeq KGL_{\text{nc}}^{\oplus 2}$. This implies that the right-hand-side of (8.28) identifies with KGL_{nc} , and so the proof is finished. \square

Lemma 8.29. *One has an isomorphism $\underline{\text{Hom}}(\mathbb{P}^1, KGL_{\text{nc}}) \simeq KGL_{\text{nc}} \oplus KGL_{\text{nc}}$.*

Proof. Similarly to Lemma 8.22, one observes that the evaluation of $\underline{\text{Hom}}(\mathbb{P}^1, KGL_{\text{nc}})$ at a k -scheme $X \in \text{Sm}'(k)$ identifies with $U_{A^1}(\text{perf}_{\text{dg}}(X \times \mathbb{P}^1))$. As explained in the proof of [49, Thm. 4.2], we have $U_{A^1}(\text{perf}_{\text{dg}}(\mathbb{P}^1)) \simeq U_{A^1}(k) \oplus U_{A^1}(k)$. By

combining this isomorphism with Remark 5.3 and with the fact that U_{A^1} is symmetric monoidal, we conclude that $U_{A^1}(\text{perf}_{\text{dg}}(X \times \mathbb{P}^1))$ is naturally isomorphic to $U_{A^1}(\text{perf}_{\text{dg}}(X)) \oplus U_{A^1}(\text{perf}_{\text{dg}}(X))$. The proof follows now from Lemma 8.22. \square

We conclude this section with the following representability result:

Proposition 8.30. *Given a k -scheme $X \in \text{Sm}'(k)$, one has isomorphisms:*

$$(8.31) \quad \underline{\text{Hom}}_{\text{M}^Q}(\Sigma^\infty(X_+), \text{KGL}_{\text{nc}}) \simeq U_{A^1}(\text{perf}_{\text{dg}}(X))$$

$$(8.32) \quad \underline{\text{Hom}}_{\mathcal{S}}(\Sigma^\infty(X_+), \text{KGL}_{\text{nc}}) \simeq KH(X).$$

Proof. Following the above adjunction (8.2), we have $(\text{KGL}_{\text{nc}})_0 = KGL_{\text{nc}}$. As a consequence, it suffices to show that $\underline{\text{Hom}}_{\text{M}^Q}(X, KGL_{\text{nc}}) \simeq U_{A^1}(\text{perf}_{\text{dg}}(X))$ and that $\underline{\text{Hom}}_{\mathcal{S}}(X, KGL_{\text{nc}}) \simeq KH(X)$. The first isomorphism follows from the Yoneda lemma combined with Lemma 8.22. The second one follows from

$$(8.33) \quad \begin{aligned} \underline{\text{Hom}}_{\mathcal{S}}(X, KGL_{\text{nc}}) &\simeq \text{Hom}_{\mathcal{S}}(U_{A^1}(k), \underline{\text{Hom}}_{\text{M}^Q}(X, KGL_{\text{nc}})) \\ &\simeq \underline{\text{Hom}}_{\mathcal{S}}(U_{A^1}(k), U_{A^1}(\text{perf}_{\text{dg}}(X))) \\ &\simeq KH(X), \end{aligned}$$

where (8.33) is a particular case of (5.2). \square

9. PROOF OF THE FIRST MAIN RESULT

In this section we prove Theorem 2.1.

Lemma 9.1. *Let \mathcal{E} and $(\mathcal{C}, \otimes, \mathbf{1})$ be monoidal categories with \mathcal{C} enriched over \mathcal{E} .*

- (i) *Assume that \mathcal{E} and \mathcal{C} are closed. In this case, one has a natural transformation $\underline{\text{Hom}}_{\mathcal{E}}(\mathbf{1}, (-)^\vee) \Rightarrow \underline{\text{Hom}}_{\mathcal{E}}(\mathbf{1}, -)^\vee$;*
- (ii) *Assume that \mathcal{C} is closed. In this case, $\underline{\text{Hom}}_{\mathcal{E}}(a \otimes b, c) \simeq \underline{\text{Hom}}_{\mathcal{E}}(a, \underline{\text{Hom}}_{\mathcal{C}}(b, c))$;*
- (iii) *Assume that \mathcal{C} is rigid. In this case, one has a natural symmetric monoidal isomorphism between lax monoidal functors $\underline{\text{Hom}}_{\mathcal{E}}(\mathbf{1}, -) \xrightarrow{\sim} \underline{\text{Hom}}_{\mathcal{E}}((-)^\vee, \mathbf{1})$.*

Proof. The proof is standard and so we leave it to the reader. \square

Comparison functor Φ . Following Ayoub [1, §4.4-4.5], consider the symmetric monoidal model category of (\mathbb{P}^1, ∞) -spectra

$$\text{SH}(k) := \text{Sp}_{(\mathbb{P}^1, \infty)}^\Sigma(L_{A^1, \text{Nis}} \text{Fun}(\text{Sm}'(k)^{\text{op}}, \mathcal{S})).$$

The above Quillen adjunction (8.24) extends naturally to

$$\begin{array}{c} \text{SH}(k; \text{M}^Q) \\ \uparrow \quad \downarrow \Gamma \\ \iota \quad \downarrow \\ \text{SH}(k). \end{array}$$

Proposition 9.2. *The (\mathbb{P}^1, ∞) -spectrum $\Gamma(\text{KGL}_{\text{nc}})$ is isomorphic to KGL .*

Proof. Recall from the proof of Theorem 8.25 that KGL_{nc} is obtained from KGL_{nc} using the general Proposition 7.1. As a consequence, we observe that $\Gamma(\text{KGL}_{\text{nc}})$ can also be obtained from $\Gamma(KGL_{\text{nc}})$ using the same general Proposition 7.1. Concretely, $\Gamma(\text{KGL}_{\text{nc}})$ can be expressed as the (\mathbb{P}^1, ∞) -spectrum associated to homotopy algebraic K -theory $KH : \text{Sm}'(k)^{\text{op}} \rightarrow \mathcal{S}, X \mapsto KH(X)$ (see Lemma 8.26) and to the element $[\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\mathbb{P}^1}(-1)] \in KH_0(\mathbb{P}^1)$. As proved in [10, §2], this is a description of KGL and so the proof is finished. \square

Remark 9.3. Thanks to the Proposition 9.2, we write KGL instead of $\Gamma(\mathrm{KGL}_{\mathrm{nc}})$.

Since $\mathrm{KGL}_{\mathrm{nc}}$ is a commutative monoid and Γ is a lax symmetric monoidal functor, KGL is also a commutative monoid. Making use of [47, Thm. 4.1], one then obtains well-defined symmetric monoidal model categories $\mathrm{Mod}(\mathrm{KGL})$ and $\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}})$ and Quillen adjunctions

$$(9.4) \quad \begin{array}{ccc} \mathrm{Mod}(\mathrm{KGL}) & & \mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}) \\ \uparrow -\wedge \mathrm{KGL} & \downarrow \text{forget} & \uparrow -\otimes \mathrm{KGL}_{\mathrm{nc}} \\ \mathrm{SH}(k) & & \mathrm{SH}(k; \mathrm{M}^Q) \end{array}$$

Note that the M^Q -enrichment of $\mathrm{SH}(k; \mathrm{M}^Q)$ extends to $\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}})$. Note also that the composition $\Gamma \circ \text{forget}$ (which preserves (trivial) fibrations) takes values in $\mathrm{Mod}(\mathrm{KGL})$. Its left adjoint is given by the left Kan extension $\bar{\iota}$ of the composition $\iota \circ (-\otimes \mathrm{KGL}_{\mathrm{nc}})$ along $-\wedge \mathrm{KGL}$. In particular, we have the following commutative square of left Quillen symmetric monoidal functors

$$(9.5) \quad \begin{array}{ccc} \mathrm{SH}(k; \mathrm{M}^Q) & \xrightarrow{-\otimes \mathrm{KGL}_{\mathrm{nc}}} & \mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}) \\ \uparrow \iota & & \uparrow \bar{\iota} \\ \mathrm{SH}(k) & \xrightarrow{-\wedge \mathrm{KGL}} & \mathrm{Mod}(\mathrm{KGL}) \end{array}$$

Finally, the triangulated comparison functor Φ is defined by the composition

$$\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL})) \xrightarrow{\bar{\iota}} \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}})) \xrightarrow{\mathrm{Hom}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, -)} \mathrm{Ho}(\mathrm{M}^Q) \simeq \mathrm{Mot}_{A^1}(k).$$

Natural transformation θ . Thanks to diagram (9.5), the functor (2.2) (which sends $X \in \mathrm{Sm}'(k)$ to $\Phi(\Sigma^\infty(X_+) \wedge \mathrm{KGL})$) admits the following description

$$(9.6) \quad \mathrm{Sm}'(k) \rightarrow \mathrm{Mot}_{A^1}(k) \quad X \mapsto \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, \Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}).$$

Lemma 9.7. *The functor (2.3) (which sends $X \in \mathrm{Sm}'(k)$ to $U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X))^\vee$) admits the following description*

$$(9.8) \quad \mathrm{Sm}'(k) \rightarrow \mathrm{Mot}_{A^1}(k) \quad X \mapsto \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, (\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^\vee)^\vee.$$

Proof. The proof is based on the following sequence of natural isomorphisms

$$\begin{aligned} & \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, (\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^\vee)^\vee \\ &= \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, \underline{\mathrm{Hom}}(\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}, \mathrm{KGL}_{\mathrm{nc}}))^\vee \\ (9.9) \quad & \simeq \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}, \mathrm{KGL}_{\mathrm{nc}})^\vee \\ (9.10) \quad & \simeq \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\Sigma^\infty(X_+), \mathrm{KGL}_{\mathrm{nc}})^\vee \stackrel{(8.31)}{\simeq} U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X))^\vee, \end{aligned}$$

where (9.9) follows from Lemma 9.1(ii) and (9.10) from adjunction (9.4). \square

Now, consider the following composition

$$\begin{aligned} & \Phi(\Sigma^\infty(X_+) \wedge \mathrm{KGL}) \stackrel{(9.6)}{\simeq} \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, \Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}) \\ (9.11) \quad & \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, \Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^{\vee\vee} \\ (9.12) \quad & \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, (\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^\vee)^\vee \\ & \stackrel{(9.8)}{\simeq} U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X))^\vee, \end{aligned}$$

where (9.11) is the canonical map to the bidual and (9.12) is the dual of the map

$\underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, (\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^\vee) \rightarrow \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, \Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^\vee$
induced by Lemma 9.1 (with $\mathcal{E} = \mathrm{M}^Q$ and $\mathcal{C} = \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))$). This composition is natural on X and hence gives rise to the desired natural transformation

$$\theta : \Phi(\Sigma^\infty(-_+) \wedge \mathrm{KGL}) \Rightarrow U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(-))^\vee.$$

Item (i). Recall from above the definition of Φ . Since τ is a symmetric monoidal functor which preserves arbitrary direct sums, it suffices to show that the functor $\underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, -)$ is lax symmetric monoidal and preserves arbitrary sums. This follows from the fact that $\mathrm{KGL}_{\mathrm{nc}}$ is the compact \otimes -unit of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))$.

Item (ii). We start by showing that the restriction of Φ to $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$, i.e.

$$\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}} \xrightarrow{\tau} \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}})) \xrightarrow{\underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, -)} \mathrm{Ho}(\mathrm{M}^Q) \simeq \mathrm{Mot}_{A^1}(k),$$

is symmetric monoidal. Let $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))^{\mathrm{pj}}$ be the thick triangulated subcategory of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))$ generated by the objects $\Sigma^\infty(X_+) \wedge \mathrm{KGL}_{\mathrm{nc}}$ with $X \in \mathrm{SmProj}(k)$. Thanks to diagram (9.5), the symmetric monoidal functor τ in the above composition takes values in $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))^{\mathrm{pj}}$. Hence, our claim follows from the following result:

Proposition 9.13. *The following functor is symmetric monoidal*

$$(9.14) \quad \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, -) : \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))^{\mathrm{pj}} \longrightarrow \mathrm{Ho}(\mathrm{M}^Q) \simeq \mathrm{Mot}_{A^1}(k).$$

Proof. Thanks to adjunction (9.4), $\underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, \mathrm{KGL}_{\mathrm{nc}})$ identifies with

$$\underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\Sigma^\infty(\mathrm{Spec}(k)_+), \mathrm{KGL}_{\mathrm{nc}}) \stackrel{(8.31)}{\simeq} U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k))) \simeq U_{A^1}(k).$$

This implies that the functor (9.14) preserves the \otimes -unit. Since by construction the category $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))^{\mathrm{pj}}$ is rigid, Lemma 9.1(iii) (with $\mathcal{E} = \mathrm{M}^Q$) furnish us a natural symmetric monoidal isomorphism

$$(9.15) \quad \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, -) \stackrel{\sim}{\simeq} \underline{\mathrm{Hom}}_{\mathrm{M}^Q}((-)^\vee, \mathrm{KGL}_{\mathrm{nc}}).$$

This implies that the functor (9.14) is symmetric monoidal if and only if the right-hand-side of (9.15) is symmetric monoidal. Since $(-)^\vee$ is symmetric monoidal, it suffices then to show that $\underline{\mathrm{Hom}}_{\mathrm{M}^Q}(-, \mathrm{KGL}_{\mathrm{nc}})$ is symmetric monoidal. Moreover, by definition of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))^{\mathrm{pj}}$, it is enough to consider the following composition (the second functor is contravariant)

$$(9.16) \quad \begin{array}{c} \mathrm{SmProj}(k) \\ \downarrow \Sigma^\infty(-_+) \otimes \mathrm{KGL}_{\mathrm{nc}} \\ \mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))^{\mathrm{pj}} \\ \downarrow \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(-, \mathrm{KGL}_{\mathrm{nc}}) \\ \mathrm{Ho}(\mathrm{M}^Q) \simeq \mathrm{Mot}_{A^1}(k). \end{array}$$

Proposition 8.30 and adjunction (9.4) imply that (9.16) identifies with

$$(9.17) \quad \mathrm{SmProj}(k)^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathrm{M}^Q) \simeq \mathrm{Mot}_{A^1}(k) \quad X \mapsto U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X)).$$

The proof follows now from the combination of Remark 5.3 with the fact that the functor U_{A^1} is symmetric monoidal. \square

Let us now show that the restriction of Φ to $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ is also fully-faithful. Since the category $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ is rigid and Φ is symmetric monoidal, it suffices then to show that the induced map of spectra

$$\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathrm{KGL}, \Sigma^\infty(X_+) \wedge \mathrm{KGL}) \rightarrow \underline{\mathrm{Hom}}_{\mathcal{S}}(U_{A^1}(k), \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, \Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}))$$

is a weak equivalence for every $X \in \mathrm{SmProj}(k)$. In what concerns the left-hand-side, we have the following weak equivalences

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{S}}(\mathrm{KGL}, \Sigma^\infty(X_+) \wedge \mathrm{KGL}) &\stackrel{(9.23)}{\simeq} \underline{\mathrm{Hom}}_{\mathcal{S}}(\mathrm{KGL}, (\Sigma^\infty(X_+) \wedge \mathrm{KGL})^\vee) \\ (9.18) \quad &\simeq \underline{\mathrm{Hom}}_{\mathcal{S}}(\Sigma^\infty(X_+) \wedge \mathrm{KGL}, \mathrm{KGL}) \\ (9.19) \quad &\simeq \underline{\mathrm{Hom}}_{\mathcal{S}}(\Sigma^\infty(X_+), \mathrm{KGL}) \simeq KH(X), \end{aligned}$$

where (9.18) follows from Lemma 9.1(ii) (with $\mathcal{E} = \mathcal{S}$) and (9.19) from adjunction (9.4). Note that we have also the following isomorphisms

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathrm{KGL}_{\mathrm{nc}}, \Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}) \\ (9.20) \quad &\stackrel{(9.24)}{\simeq} \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\mathrm{KGL}_{\mathrm{nc}}, (\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^\vee) \\ &\simeq \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}, \mathrm{KGL}_{\mathrm{nc}}) \\ (9.21) \quad &\simeq \underline{\mathrm{Hom}}_{\mathrm{M}^Q}(\Sigma^\infty(X_+), \mathrm{KGL}_{\mathrm{nc}}) \stackrel{(8.31)}{\simeq} U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X)), \end{aligned}$$

where (9.20) follows from Lemma 9.1(ii) (with $\mathcal{E} = \mathrm{M}^Q$) and (9.21) from adjunction (9.4). As a consequence, the above induced map of spectra reduces to $KH(X) \rightarrow \underline{\mathrm{Hom}}_{\mathcal{S}}(U_{A^1}(k), U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X)))$. The proof follows now from isomorphism (5.2).

Lemma 9.22. *For every $X \in \mathrm{SmProj}(k)$, one has canonical duality isomorphisms*

$$\begin{aligned} (9.23) \quad &\Sigma^\infty(X_+) \wedge \mathrm{KGL} \simeq (\Sigma^\infty(X_+) \wedge \mathrm{KGL})^\vee \\ (9.24) \quad &\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}} \simeq (\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}})^\vee. \end{aligned}$$

Proof. Isomorphism (9.23) is obtained by combining [13, Thm. 5.23 and Example 2.12] with isomorphism $\mathrm{KGL} \simeq \mathrm{KGL}(1)[2]$ (see [11, §13.2]). Isomorphism (9.24) is obtained from (9.23) by applying the symmetric monoidal functor $\bar{\iota}$. \square

Item (iii). Since by assumption $\Sigma^\infty(X_+) \wedge \mathrm{KGL}$ belongs to $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$, diagram (9.5) implies that $\Sigma^\infty(X_+) \otimes \mathrm{KGL}_{\mathrm{nc}}$ belongs to $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathrm{nc}}))^{\mathrm{pj}}$. Since this latter category is rigid and the above functor (9.14) is symmetric monoidal, we then conclude that the maps (9.11)-(9.12) are isomorphisms. This implies that the natural transformation θ is an isomorphism at the k -scheme X .

Item (iv). Let $X \in \mathrm{SmProj}(k)$. As explained in §4.2-4.3, the dg category $\mathrm{perf}_{\mathrm{dg}}(X)$ is smooth and proper in the sense of Kontsevich. This implies that $U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X))^\vee$ belongs to the rigid category $\mathrm{KMM}_{A^1}(k)$. Using the fact that θ is an isomorphism at every smooth projective k -scheme, one then concludes that the restriction of Φ to $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ takes values in $\mathrm{KMM}_{A^1}(k) \subset \mathrm{Mot}_{A^1}(k)$.

Now, let us denote by $\mathrm{KMM}(k)^{\mathrm{pj}}$ (resp. $\mathrm{KMM}_{A^1}(k)^{\mathrm{pj}}$) be the thick triangulated subcategory of $\mathrm{KMM}(k)$ (resp. $\mathrm{KMM}_{A^1}(k)$) generated by the objects $U(\mathrm{perf}_{\mathrm{dg}}(X))^\vee$ (resp. $U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X))^\vee$). We claim that the canonical functor $\mathrm{KMM}(k)^{\mathrm{pj}} \rightarrow \mathrm{KMM}_{A^1}(k)^{\mathrm{pj}}$ is an equivalence. Since these categories are rigid, it

suffices to show that the induced map of spectra

$$(9.25) \quad \begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathcal{S}}(U(k), U(\mathrm{perf}_{\mathrm{dg}}(X))^{\vee}) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{S}}(U_{A^1}(k), U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X))^{\vee}) \\ \simeq \downarrow & & \downarrow \simeq \\ \underline{\mathrm{Hom}}_{\mathcal{S}}(U(\mathrm{perf}_{\mathrm{dg}}(X)), U(k)) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{S}}(U_{A^1}(\mathrm{perf}_{\mathrm{dg}}(X)), U_{A^1}(k)) \end{array}$$

is a weak equivalence for every $X \in \mathrm{SmProj}(k)$. Making use of isomorphisms (5.1)-(5.2), one observes that (9.25) identifies with the canonical map $\mathcal{K}(X) \rightarrow KH(X)$ from nonconnective algebraic K -theory to the homotopy algebraic K -theory. The proof follows now from the fact that these algebraic K -theories are the same since by assumption X is smooth; see [63, §6].

Remark 9.26. By combining (9.23) with Theorem 2.1, one obtains a duality isomorphism $U(\mathrm{perf}_{\mathrm{dg}}(X))^{\vee} \simeq U(\mathrm{perf}_{\mathrm{dg}}(X))$ for every $X \in \mathrm{SmProj}(k)$.

Proof of Corollary 2.4. Item (i). Let us denote $\mathrm{Ho}(\mathrm{SH}(k))^{\mathrm{pj}}$ the thick triangulated subcategory of $\mathrm{Ho}(\mathrm{SH}(k))$ generated by the objects $\Sigma^{\infty}(X_+)(i)$ with $X \in \mathrm{SmProj}(k)$ and $i \in \mathbb{Z}$. As proved in [1, Prop. 2.2.27-1] (see also [43]), $\mathrm{Ho}(\mathrm{SH}(k))^{\mathrm{pj}}$ agrees with the category of compact objects of $\mathrm{Ho}(\mathrm{SH}(k))$. Using the isomorphism $\mathrm{KGL} \simeq \mathrm{KGL}(1)[2]$ (see [11, §13.2]) and the adjunction (9.4), one then concludes that $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ agrees with the category of compact objects of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))$. Note that for every k -scheme $X \in \mathrm{Sm}'(k)$, the KGL -module $\Sigma^{\infty}(X_+) \wedge \mathrm{KGL}$ is compact. Hence, Theorem 2.1 implies that the first diagram of Corollary 2.4 is commutative. Let us study the comparison functor Φ . Thanks to Theorem 2.1, Φ is lax symmetric monoidal, preserves arbitrary sums, and its restriction to $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ is fully-faithful, symmetric monoidal, and takes values in $\mathrm{KMM}(k)$. Since $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))^{\mathrm{pj}}$ agrees with the category of compact objects of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))$, one concludes that Φ is fully-faithful, symmetric monoidal (since the symmetric monoidal structures of $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))$ and $\mathrm{Mot}_{A^1}(k)$ are homotopy colimit preserving), and that it takes values in $\mathrm{KMM}^{\oplus}(k)$. This achieves the proof.

Item (ii). Since Theorem 2.1 holds also with \mathbb{Q} -coefficients, the proof is similar to item (i); simply replace [1, Prop. 2.2.27-1] by [1, Prop. 2.2.27-2] or use instead [43].

Item (iii). As explained above, the triangulated categories $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}))$ and $\mathrm{Ho}(\mathrm{Mod}(\mathrm{KGL}_{\mathbb{Q}}))$ are compactly generated and the triangulated functors Φ and $\Phi_{\mathbb{Q}}$ preserves arbitrary sums. Hence, the existence of the right adjoints Ψ and $\Psi_{\mathbb{Q}}$ follows from [40, Thm. 8.44].

Item (iv). As explained above, the triangulated category $\mathrm{Ho}(\mathrm{SH}(k))$ is compactly generated by the objects $\Sigma^{\infty}(X_+)(i)[n]$ with $X \in \mathrm{SmProj}(k)$ and $i, n \in \mathbb{Z}$. Thanks to the construction of K^A and the isomorphism $K^A \simeq K^A(1)[2]$ (see [25, §5-6]), we have natural isomorphisms

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{SH}(k))}(\Sigma^{\infty}(X_+)(i)[n], K^A) \\ \simeq & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{SH}(k))}(\Sigma^{\infty}(X_+), K^A[-n+2i]) \simeq K_{n-2i}(X; A), \end{aligned}$$

where $K_{n-2i}(X; A)$ stands for the $(n-2i)^{\mathrm{th}}$ algebraic K -theory group of the exact category of $\mathcal{O}_X \otimes A$ -modules which are locally free and of finite rank as \mathcal{O}_X -modules.

On the other hand, we have the following natural isomorphisms:

$$\begin{aligned}
(9.27) \quad & \text{Hom}_{\text{Ho}(\text{SH}(k))}(\Sigma^\infty(X_+)(i)[n], \Psi(U(A))) \\
& \simeq \text{Hom}_{\text{Ho}(\text{SH}(k))}(\Sigma^\infty(X_+) \wedge \text{KGL}, \Psi(U(A)))[-n+2i]) \\
(9.28) \quad & \simeq \text{Hom}_{\text{KMM}^\oplus(k)}(U(\text{perf}_{\text{dg}}(X))^\vee, U(A)[-n+2i]) \\
(9.29) \quad & \simeq \text{Hom}_{\text{KMM}^\oplus(k)}(U(k), U(\text{perf}_{\text{dg}}(X) \otimes (A))[-n+2i]) \\
(9.30) \quad & \simeq K_{n-2i}(\text{perf}_{\text{dg}}(X) \otimes A) \\
(9.31) \quad & \simeq K_{n-2i}^A(X; A).
\end{aligned}$$

Some explanations are in order: (9.27) follows from adjunction (9.4) and isomorphism $\text{KGL} \simeq \text{KGL}(1)[2]$; (9.28) follows from the adjunction (Φ, Ψ) and the commutative diagram of item (i); (9.29) follows from the fact that U is symmetric monoidal and that $U(\text{perf}_{\text{dg}}(X))$ is a strongly dualizable object (since the dg category $\text{perf}_{\text{dg}}(X)$ is smooth and proper); (9.30) follows from the co-representability of algebraic K -theory (see [7, Thm. 7.16]); and (9.31) follows from [56, Lems. 6.2 and 6.4] and their proofs. The above natural isomorphisms imply that K^A and $\Psi(U(A))$ represent the same functor in the homotopy category $\text{Ho}(\text{SH}(k))$. Hence, making use of the Yoneda lemma, we conclude that they are isomorphic.

Remark 9.32. The proofs of Theorem 2.1 and of items (i)-(iii) of Corollary 2.4 work *mutatis mutandis* the same for every regular ring k : simple replace [17, 44] by [11, §13.3] in §2; $-\otimes-$ by $-\otimes^{\mathbf{L}}-$ in §4.2; and remove [43] in the proof of Corollary 2.4.

10. PROOF OF THE SECOND MAIN RESULT

In this section we prove Theorem 2.7. By construction, the comparison functor \mathbf{R} sends $\mathbb{Q}(1)[2]$ to the \otimes -unit of $\text{KMM}(k)_{\mathbb{Q}}$. As a consequence, using the universal property of the projection functor π (see §4.4), one obtains a well-defined \mathbb{Q} -linear additive symmetric monoidal functor $\bar{\mathbf{R}}$ making diagram (2.8) commute. Let us now show that $\bar{\mathbf{R}}$ is also fully-faithful. As proved in [62, Cor. 3.5.5] (when k is of characteristic zero) and in [28, Prop. 5.5.3] (when k is of positive characteristic), the following set generates the triangulated category $\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$

$$\mathcal{G} := \{M(X)_{\mathbb{Q}}[-n] \mid X \in \text{SmProj}(k) \text{ and } n \in \mathbb{Z}\}.$$

Hence, making use of Lemma 10.7 below, it suffices to show that the induced map

$$(10.1) \quad \text{Hom}(\pi(M(X)_{\mathbb{Q}}), \pi(M(Y)_{\mathbb{Q}}[-n])) \longrightarrow \text{Hom}(\mathbf{R}(M(X)_{\mathbb{Q}}), \mathbf{R}(M(Y)_{\mathbb{Q}})[-n])$$

is an isomorphism for any two $X, Y \in \text{SmProj}(k)$ and $n \in \mathbb{Z}$. In what concerns the left-hand-side, we have the following isomorphisms ($d := \dim(Y)$):

$$\begin{aligned}
& = \oplus_{i \in \mathbb{Z}} \text{Hom}_{\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}}(M(X)_{\mathbb{Q}}, M(Y)_{\mathbb{Q}}(i)[2i-n]) \\
(10.2) \quad & \simeq \oplus_{i \in \mathbb{Z}} \text{Hom}_{\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}}(M(X \times Y)_{\mathbb{Q}}, \mathbb{Q}(i+d)[2(i+d)-n]) \\
(10.3) \quad & \simeq \oplus_{i \in \mathbb{Z}} H^{2(i+d)-n}(X \times Y, \mathbb{Q}(i+d)) \\
(10.4) \quad & \simeq \oplus_{i \in \mathbb{Z}} CH^{(i+d)}(X \times Y, n)_{\mathbb{Q}} \simeq \oplus_{i \in \mathbb{Z}} CH^i(X \times Y, n)_{\mathbb{Q}}.
\end{aligned}$$

Some explanations are in order: (10.2) follows from the fact that $M(Y)_{\mathbb{Q}}(-d)[-2d]$ is the strong dual of $M(Y)_{\mathbb{Q}}$ (see [2, §18.4]); (10.3) follows from the representability of motivic cohomology in $\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$; and (10.4) follows from the identification

between motivic cohomology and Bloch's higher Chow groups (see [61]). In what concerns the right-hand-side, we have the following isomorphisms

$$\begin{aligned}
 &= \operatorname{Hom}_{\operatorname{KMM}(k)_{\mathbb{Q}}}(U(\operatorname{perf}_{\operatorname{dg}}(X))_{\mathbb{Q}}^{\vee}, U(\operatorname{perf}_{\operatorname{dg}}(Y))_{\mathbb{Q}}^{\vee}[-n]) \\
 (10.5) \quad &\simeq \operatorname{Hom}_{\operatorname{KMM}(k)_{\mathbb{Q}}}(U(\operatorname{perf}_{\operatorname{dg}}(X))_{\mathbb{Q}}, U(\operatorname{perf}_{\operatorname{dg}}(Y))_{\mathbb{Q}}[-n]) \\
 (10.6) \quad &\simeq \mathcal{H}(X \times Y)_{\mathbb{Q}} \simeq K_n(X \times Y)_{\mathbb{Q}},
 \end{aligned}$$

where (10.5) follows from Remark (9.26) and (10.6) from weak equivalence (5.1). The proof follows now from the fact that (10.1) identifies with the classical isomorphism $\oplus_{i \in \mathbb{Z}} CH^i(X \times Y, n)_{\mathbb{Q}} \simeq K_n(X \times Y)_{\mathbb{Q}}$; see [35].

Lemma 10.7. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated category, $O \in \mathcal{C}$ a \otimes -invertible object, $R : \mathcal{C} \rightarrow \mathcal{D}$ a symmetric monoidal triangulated functor which sends O to the \otimes -unit of \mathcal{D} , and*

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{R} & \mathcal{D} \\
 \pi \downarrow & \nearrow \overline{R} & \\
 \mathcal{C}/_{\perp \otimes O} & &
 \end{array}$$

the induced commutative diagram. Assume that \mathcal{C} has a set of generators \mathcal{G} and that the restriction of \overline{R} to the full subcategory $\pi\mathcal{G} \subset \mathcal{C}/_{\perp \otimes O}$ is fully-faithful. Under these assumptions, the functor \overline{R} is fully-faithful.

Proof. The triangulated category \mathcal{C} admits a canonical filtration

$$\mathcal{G} =: \langle \mathcal{G} \rangle_0 \subset \langle \mathcal{G} \rangle_1 \subset \cdots \subset \langle \mathcal{G} \rangle_n \subset \langle \mathcal{G} \rangle_{n+1} \subset \cdots \subset \mathcal{C},$$

where $\langle \mathcal{G} \rangle_{n+1}$ is the category of those objects b appearing in a distinguished triangle

$$(10.8) \quad a \longrightarrow b \longrightarrow c \longrightarrow a[1]$$

with $a \in \langle \mathcal{G} \rangle_n$ and $c \in \langle \mathcal{G} \rangle_0$. Let us write $\pi\langle \mathcal{G} \rangle_n$ for the full subcategory of $\mathcal{C}/_{\perp \otimes O}$ with the same objects as $\langle \mathcal{G} \rangle_n$ and $\{\pi\langle \mathcal{G} \rangle_n\}_{n \geq 0}$ for the associated filtration. Note that $\mathcal{C} = \cup_n \langle \mathcal{G} \rangle_n$ and $\mathcal{C}/_{\perp \otimes O} = \cup_n \pi\langle \mathcal{G} \rangle_n$. By assumption, $\overline{R}|_{\pi\langle \mathcal{G} \rangle_0}$ is fully-faithful. Hence, let us prove that if by hypothesis $\overline{R}|_{\pi\langle \mathcal{G} \rangle_n}$ is fully-faithful, then $\overline{R}|_{\pi\langle \mathcal{G} \rangle_{n+1}}$ is also fully-faithful. Given any object $d \in \mathcal{C}$, the definition of $\mathcal{C}/_{\perp \otimes O}$ allow us to conclude that (10.8) gives rise to long exact sequences

$$\begin{aligned}
 &\cdots \operatorname{Hom}_{\mathcal{C}/_{\perp \otimes O}}(\pi(d), \pi(a)) \rightarrow \operatorname{Hom}_{\mathcal{C}/_{\perp \otimes O}}(\pi(d), \pi(b)) \rightarrow \operatorname{Hom}_{\mathcal{C}/_{\perp \otimes O}}(\pi(d), \pi(c)) \cdots \\
 &\cdots \operatorname{Hom}_{\mathcal{C}/_{\perp \otimes O}}(\pi(c), \pi(d)) \rightarrow \operatorname{Hom}_{\mathcal{C}/_{\perp \otimes O}}(\pi(b), \pi(d)) \rightarrow \operatorname{Hom}_{\mathcal{C}/_{\perp \otimes O}}(\pi(a), \pi(d)) \cdots
 \end{aligned}$$

The proof follows now from a simple application of the classical 5-lemma. \square

11. REMAINING PROOFS

Proof of Proposition 2.9. Let us start by constructing V_{nc} . Since a triangulated category is additive and the restriction of a localizing invariant to its base category is an additive invariant (see [53, §13]), we conclude that

$$(11.1) \quad \operatorname{dgc}at(k) \longrightarrow \operatorname{Ho}(\operatorname{dgc}at(k)) \xrightarrow{U(-)_{\mathbb{Q}}} \operatorname{Mot}(k)_{\mathbb{Q}}$$

is an additive invariant. Making use of equivalence (5.4), one then obtains a well-defined \mathbb{Q} -linear additive functor V_{nc} making the following diagram commute

$$(11.2) \quad \begin{array}{ccc} \text{dgc}at(k) & \xrightarrow{\quad} & \text{Ho}(\text{dgc}at(k)) \\ U_{\text{add}}(-)_{\mathbb{Q}} \downarrow & & \downarrow U(-)_{\mathbb{Q}} \\ \text{Hmo}_0(k)_{\mathbb{Q}} & \xrightarrow{V_{\text{nc}}} & \text{Mot}(k)_{\mathbb{Q}}. \end{array}$$

Moreover, since (11.1) is symmetric monoidal, [9, Prop. 5.5] implies that the functor V_{nc} is also symmetric monoidal. Now, recall from [54, §6][8, Lem. 5.9 and Thm. 9.2] that we have the following isomorphisms

$$\text{Hom}_{\text{Hmo}_0(k)_{\mathbb{Q}}}(U_{\text{add}}(\mathcal{A})_{\mathbb{Q}}, U_{\text{add}}(\mathcal{B})_{\mathbb{Q}}) \simeq K_0(\mathcal{A}^{\text{op}} \otimes \mathcal{B})_{\mathbb{Q}} \simeq \text{Hom}_{\text{Mot}(k)_{\mathbb{Q}}}(U(\mathcal{A})_{\mathbb{Q}}, U(\mathcal{B})_{\mathbb{Q}})$$

for any two dg categories \mathcal{A}, \mathcal{B} with \mathcal{A} smooth and proper. Consequently, we conclude that the restricted functor $V_{\text{nc}} : \text{NChow}(k)_{\mathbb{Q}} \rightarrow \text{KMM}(k)_{\mathbb{Q}}$ is also fully-faithful. It remains then only to show that the right-hand-side square of diagram (2.10) is commutative (up to isomorphism). Recall that every object of $\text{Chow}(k)_{\mathbb{Q}}$ is a direct factor of a Chow motive of the form $h(X)_{\mathbb{Q}}(m)$, with $X \in \text{SmProj}(k)$ and $m \in \mathbb{Z}$. Since the categories $\text{Chow}(k)_{\mathbb{Q}}$ and $\text{Chow}(k)_{\mathbb{Q}}/\otimes_{\mathbb{Q}(1)}$ have the same objects and $\pi(h(X)_{\mathbb{Q}}(m)) \simeq \pi(h(X)_{\mathbb{Q}})$, it suffices to treat the objects of the form $\pi(h(X)_{\mathbb{Q}})$. Let us start by showing that the objects

$$(11.3) \quad V_{\text{nc}}(\overline{R}(\pi(h(X)_{\mathbb{Q}})))^{\vee} \quad \text{and} \quad \overline{R}(\overline{V}(\pi(h(X)_{\mathbb{Q}})))$$

are isomorphic. As proved in [50, Thm. 1.1], $\overline{R}(\pi(h(X)_{\mathbb{Q}})) \simeq U_{\text{add}}(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}$. Hence, making use of the above diagram (11.2), one observes that the left-hand-side of (11.3) identifies with $U(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}^{\vee}$. On the other hand, since $\overline{V}(\pi(h(X)_{\mathbb{Q}})) \simeq M(X)_{\mathbb{Q}}$, Theorem 2.1 implies that the right-hand-side of (11.3) also identifies with $U(\text{perf}_{\text{dg}}(X))^{\vee}$. Given smooth projective k -schemes X and Y , let us now show the following diagram commutes (up to isomorphism)

$$\begin{array}{ccc} \text{Hom}(\pi(h(X)_{\mathbb{Q}}), \pi(h(Y)_{\mathbb{Q}})) & \xrightarrow{\overline{R}} & \text{Hom}(U_{\text{add}}(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}, U_{\text{add}}(\text{perf}_{\text{dg}}(Y))_{\mathbb{Q}}) \\ \downarrow \overline{V} & & \downarrow V_{\text{nc}} \\ & & \text{Hom}(U(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}, U(\text{perf}_{\text{dg}}(Y))_{\mathbb{Q}}) \\ & & \downarrow (-)^{\vee} \\ \text{Hom}(\pi(M(Y)_{\mathbb{Q}}), \pi(M(X)_{\mathbb{Q}})) & \xrightarrow{\overline{R}} & \text{Hom}(U(\text{perf}_{\text{dg}}(Y))^{\vee}_{\mathbb{Q}}, U(\text{perf}_{\text{dg}}(X))^{\vee}_{\mathbb{Q}}). \end{array}$$

As explained in the proof of [50, Thm. 1.1], \overline{R} is given by the classical isomorphism $\oplus_{i \in \mathbb{Z}} CH^i(X \times Y)_{\mathbb{Q}} \simeq K_0(X \times Y)_{\mathbb{Q}}$. On the other hand, as explained in the proof of Theorem 2.7, \overline{R} is also given by the classical isomorphism $\oplus_{i \in \mathbb{Z}} CH^i(Y \times X)_{\mathbb{Q}} \simeq K_0(Y \times X)_{\mathbb{Q}}$. Since Voevodsky's isomorphism \overline{V} corresponds to the switch of X and Y , the above diagram identifies then with

$$\begin{array}{ccc} \oplus_{i \in \mathbb{Z}} CH^i(X \times Y)_{\mathbb{Q}} & \xrightarrow{\quad} & K_0(X \times Y)_{\mathbb{Q}} \\ \downarrow & & \parallel \\ & & K_0(X \times Y)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \oplus_{i \in \mathbb{Z}} CH^i(Y \times X)_{\mathbb{Q}} & \xrightarrow{\quad} & K_0(Y \times X)_{\mathbb{Q}}. \end{array}$$

This latter diagram is commutative and so the proof is finished.

Proof of Proposition 3.1. Item(i). The implication (\Rightarrow) follows from the fact that the comparison functor R is symmetric monoidal. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a \mathbb{Q} -linear idempotent complete rigid symmetric monoidal category such that $\text{End}_{\mathcal{C}}(\mathbf{1})$ has no nontrivial idempotents. Under these assumptions, it was proved in [30, Props. 8.2.6 and 8.2.9] that an object $c \in \mathcal{C}$ is \otimes -invertible if and only if $\wedge^2(c) = 0$ or $\text{Sym}^2(c) = 0$. Recall that $\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$ and $\text{KMM}(k)_{\mathbb{Q}}$ are \mathbb{Q} -linear idempotent complete rigid symmetric monoidal categories, that we have the following \mathbb{Q} -algebra isomorphisms

$$\text{End}_{\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}}(\mathbb{Q}(0)) \simeq \mathbb{Q} \quad \text{End}_{\text{KMM}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}) \simeq K_0(k) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q},$$

and that the comparison functor R is \mathbb{Q} -linear, faithful, and symmetric monoidal. As a consequence, the proof of implication (\Leftarrow) follows now from Proposition 3.10(i).

Item (ii). Thanks to Theorem 2.7, one has an injective group homomorphism

$$\text{Pic}(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}} / \text{--} \otimes_{\mathbb{Q}(1)[2]}) \hookrightarrow \text{Pic}(\text{KMM}(k)_{\mathbb{Q}}).$$

Hence, in order to prove that the group homomorphism (3.2) is injective, it suffices to show the following claim: *if a geometric mixed motive $M \in \text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$ becomes isomorphic in the orbit category to the \otimes -unit $\pi(\mathbb{Q}(0))$, then $M \in \{\mathbb{Q}(i)[2i] \mid i \in \mathbb{Z}\}$.* Let us then assume the existence of morphisms

$$\underline{f} = \{f_i\}_{i \in \mathbb{Z}} \in \oplus_{i \in \mathbb{Z}} \text{Hom}(M, \mathbb{Q}(i)[2i]) \quad \underline{g} = \{g_i\}_{i \in \mathbb{Z}} \in \oplus_{i \in \mathbb{Z}} \text{Hom}(\mathbb{Q}, M(i)[2i])$$

verifying the equalities $\underline{g} \circ \underline{f} = \text{id}$ and $\underline{f} \circ \underline{g} = \text{id}$. By definition of the orbit category (see §4.4), the following composition

$$M \xrightarrow{\{f_i\}_{i \in \mathbb{Z}}} \oplus_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i] \xrightarrow{\{g_{-i(i)[2i]}\}_{i \in \mathbb{Z}}} M$$

agrees with the 0^{th} -component of $\underline{g} \circ \underline{f}$, i.e. with the identity of M . As a consequence, M is a direct factor of $\oplus_{i \in \mathbb{Z}} \mathbb{Q}(i)[2i]$. By combining the isomorphisms

$$\text{Hom}_{\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}}(\mathbb{Q}(i)[2i], \mathbb{Q}(j)[2j]) \simeq \begin{cases} \mathbb{Q} & i = j \\ 0 & i \neq j \end{cases}$$

with the fact that $\pi(M) \simeq \pi(\mathbb{Q}(0))$, we conclude then that $M \in \{\mathbb{Q}(i)[2i] \mid i \in \mathbb{Z}\}$.

Proof of Proposition 3.8. Item (i). Recall first from (5.1) that we have the following isomorphisms

$$(11.4) \quad \text{Hom}_{\text{KMM}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, U(k)_{\mathbb{Q}}[-n]) \simeq \begin{cases} K_n(k) \otimes_{\mathbb{Z}} \mathbb{Q} & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

Now, consider the following full subcategory $\mathcal{H} := \{U(k)_{\mathbb{Q}}^{\oplus n} \mid n \geq 1\} \subset \text{KMM}(k)_{\mathbb{Q}}$. Thanks to (11.4) (with $n = 0$), one observes that \mathcal{H} is equivalent to the category of \mathbb{Q} -vector spaces. In particular, \mathcal{H} is additive, idempotent complete, and symmetric monoidal. Making use of [6, Thm. 4.3.2 II and Prop. 5.2.2], one then concludes that there exists a bounded weight structure w on the triangulated category $\langle U(k)_{\mathbb{Q}} \rangle$ with heart \mathcal{H} . Consequently, [6, Thm. 5.3.1 and Rk. 5.3.2] imply that the inclusion $\mathcal{H} \subset \langle U(k)_{\mathbb{Q}} \rangle$ gives rise to a ring isomorphism $K_0(\mathcal{H}) \simeq K_0(\langle U(k)_{\mathbb{Q}} \rangle)$. The proof follows now automatically from the canonical isomorphism $K_0(\mathcal{H}) \simeq \mathbb{Q}$.

Item (ii). The proof follows from the fact that the comparison functor (3.5) sends $\mathbb{Q}(1)$ to $U(k)_{\mathbb{Q}}[-2]$ and that $[U(k)_{\mathbb{Q}}[-2]] = [U(k)_{\mathbb{Q}}] = 1$ in $K_0(\langle U(k)_{\mathbb{Q}} \rangle)$.

Proof of Proposition 3.10. As proved in [37, Lem. 1.11], every \mathbb{Q} -linear additive symmetric monoidal functor preserves Schur-finiteness (and hence Kimura-finiteness). If the functor is moreover faithful, then it also reflects Schur-finiteness. The proof follows from these general results applied to the comparison functor R .

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